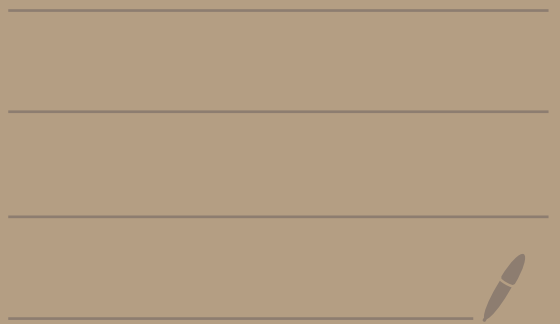


Math 3450

2/1/23



Def: Let A and B be sets.

The Cartesian product of
 A and B is

$$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

Note: (a, b) is called an ordered pair. Order matters for (a, b)

People have proposed various set definitions for (a, b) . For example one is $(a, b) = \{ a, \{ a, b \} \}$

Ex: $A = \{1, 5, 9\}$

$$B = \{4, 9\}$$

$$A \times B = \{(1, 4), (1, 9), (5, 4), (5, 9), (9, 4), (9, 9)\}$$

$$B \times B = \{(4, 4), (4, 9), (9, 4), (9, 9)\}$$

Note: In general, if S and T are finite sets, then

$$\underbrace{|S \times T|}_{\text{means size of } S \times T} = \underbrace{|S|}_{\text{size of } S} \cdot \underbrace{|T|}_{\text{size of } T}$$

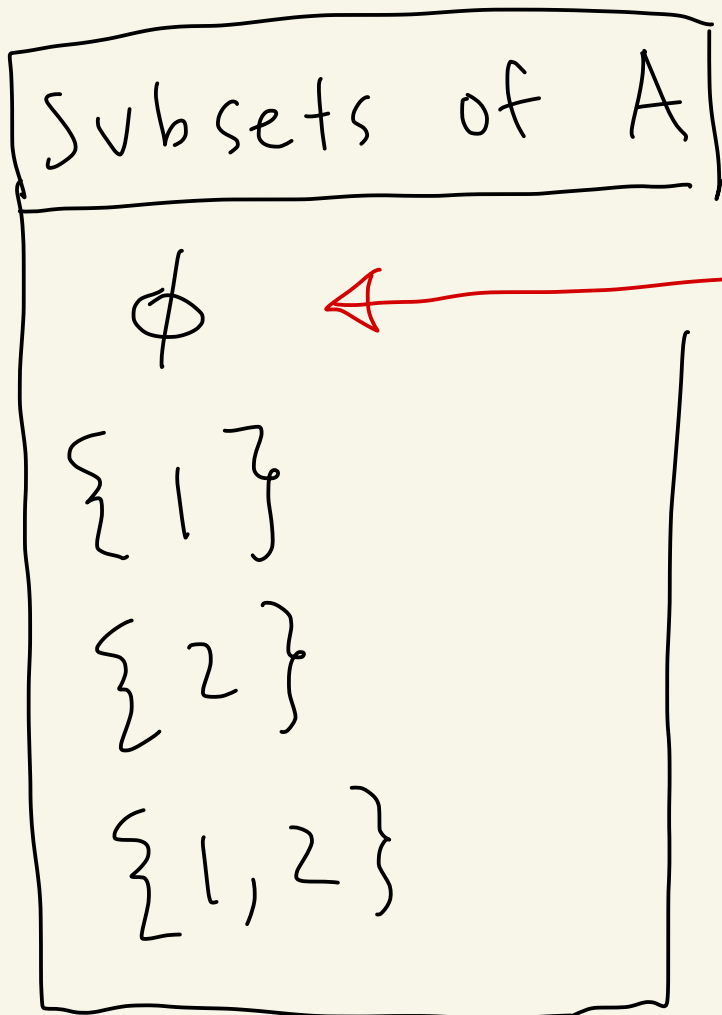
Def: Let A be a set.

We define the power set of A to be the set of all subsets of A ,

that is

$$\underbrace{P(A)}_{\text{power set of } A} = \underbrace{\{ B \mid B \subseteq A \}}_{\text{the set of all } B \text{ where } B \subseteq A}$$

Ex: $A = \{1, 2\}$



empty set is a subset of every set

$$\emptyset = \{ \}$$

SIDE COMMENTARY

$S \subseteq T$ means:

$\forall x$ (If $x \in S$, then $x \in T$)

$\emptyset \subseteq T$ means:

$(\forall x) \underbrace{(\text{If } x \in \emptyset, \text{ then } x \in T)}_F$
T

$$|P(A)| = 4 = 2^2 = 2^{|A|}$$

$$P(A) = \{ \emptyset, \{1\}, \{2\}, \{1, 2\} \}$$

Ex: $B = \{5, 2, 1\}$

$$\mathcal{P}(B) = \left\{ \emptyset, \{1\}, \{2\}, \{5\}, \right. \\ \left. \{5, 2\}, \{2, 1\}, \right. \\ \left. \{5, 1\}, \{5, 2, 1\} \right\}$$

Note: $|\mathcal{P}(B)| = 8 = 2^3 = 2^{|B|}$

Theorem: If S is finite,
then $|\mathcal{P}(S)| = 2^{|S|}$

Theorem: Let A and B
be sets. Then, $A = B$
if and only if $P(A) = P(B)$.

proof:

(\Rightarrow) It's clear that if
 $A = B$, then $P(A) = P(B)$.

(\Leftarrow) Now we must prove
"If $P(A) = P(B)$, then $A = B$ ".

Suppose $P(A) = P(B)$.

To show that $A = B$ we

will show $A \subseteq B$ and $B \subseteq A$.

Claim 1: $A \subseteq B$

We know $A \subseteq A$.

So, $A \in \mathcal{P}(A)$.

Then, since $\mathcal{P}(A) = \mathcal{P}(B)$,
we know $A \in \mathcal{P}(B)$.

Thus, $A \subseteq B$.

Claim 2: $B \subseteq A$

You can do this proof the same
way as claim 1, but let's
change it up.

Let $b \in B$.

Then, $\{b\} \subseteq B$.

So, $\{b\} \in \mathcal{P}(B)$

Since $\mathcal{P}(B) = \mathcal{P}(A)$ we have $\{b\} \in \mathcal{P}(A)$

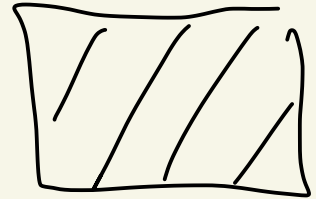
Thus, $\{b\} \subseteq A$.

Hence $b \in A$.

So, $B \subseteq A$.



By claim 1 and 2 we
know $A = B$.



HW 2 #3

Let A, B, C be sets.

Prove: If $A \subseteq B$,

then $A - C \subseteq B - C$.

proof: Assume $A \subseteq B$.

Let's prove this implies
that $A - C \subseteq B - C$.

Let $x \in A - C$.

Then, $x \in A$ and $x \notin C$.

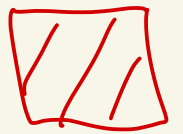
Since $x \in A$ and $A \subseteq B$,

we know that $x \in B$.

So, $x \in B$ and $x \notin C$. 

Thus, $x \in B - C$.

We have shown that $A - C \subseteq B - C$.



Q: Is the converse true? I.e.,

If $A - C \subseteq B - C$, then $A \subseteq B$.

No.

$$A = \{1, 2\}$$

$$B = \{4, 2\}$$

$$C = \{1, 4\}$$

$$A - C = \{2\}$$

$$B - C = \{2\}.$$

So, $A - C \subseteq B - C$, but $A \not\subseteq B$.

Recall:

Statement: If P , then Q .

Converse: If Q , then P .

Contrapositive: If $\neg Q$, then $\neg P$.

Def: When every element of a set A is itself a set then we call A a family or collection of sets.

Ex: $\mathcal{P}(A)$ is a family of sets.

HW 2 -

④ Let A, B be sets.

Prove that $A \subseteq B$ iff $A - B = \emptyset$.

Proof:

(\Rightarrow) Suppose $A \subseteq B$.

We must show that $A - B = \emptyset$.

What would happen if $A - B \neq \emptyset$?

Then there would exist $x \in A - B$.

Then $x \in A$ and $x \notin B$.

But then $A \not\subseteq B$.

So we must have $A - B = \emptyset$.

(\Leftarrow) Suppose $A - B = \emptyset$.

contradicts
 $A \subseteq B$

We must show $A \subseteq B$.

Pick some $a \in A$.

We must show $a \in B$.

What if $a \notin B$?

Then, $A - B \neq \emptyset$ because $a \in A - B$.

Contradiction.

Thus, $a \in B$.

Therefore, $A \subseteq B$.

