Math 3450
2/1/23

Def: Let $A$ and $B$ be sets. The Cartesian product of $A$ and $B$ is

$$A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$$

$$\boxed{\text{Note: } (a,b) \text{ is called an ordered pair. Order matters for } (a,b)}$$

People have proposed various set definitions for $(a,b)$. For example one is $(a,b) = \{ a, \{a,b\} \}$

Ex: $A = \{1, 5, 9\}$

$B = \{4, 9\}$

$$A \times B = \{(1,4), (1,9), (5,4), (5,9), (9,4), (9,9)\}$$

$$B \times B = \{(4,4), (4,9), (9,4), (9,9)\}$$

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Note: In general, if $S$ and $T$ are finite sets, then

$$\underbrace{|S \times T|}_{\text{means size of } S \times T} = \underbrace{|S|}_{\text{size of } S} \cdot \underbrace{|T|}_{\text{size of } T}$$

Def: Let $A$ be a set. We define the power set of $A$ to be the set of all subsets of $A$, that is

$$\underbrace{\mathcal{P}(A)}_{\text{power set of } A} = \underbrace{\{B \mid B \subseteq A\}}_{\text{the set of all } B \text{ where } B \subseteq A}$$

$\underline{\text{Ex:}}$ $A = \{1, 2\}$

| Subsets of $A$ |
|---|
| $\emptyset$ |
| $\{1\}$ |
| $\{2\}$ |
| $\{1,2\}$ |

empty set is a subset of every set

$\emptyset = \{\ \}$

SIDE COMMENTARY

$S \subseteq T$ means:

$\forall x$ (If $x \in S$, then $x \in T$)

$\emptyset \subseteq T$ means:

$$\underbrace{(\forall x)(\text{If } \underbrace{x \in \emptyset}_{F}, \text{ then } x \in T)}_{T}$$

$|\mathcal{P}(A)| = 4$
$= 2^2 = 2^{|A|}$

$\mathcal{P}(A) = \{ \emptyset, \{1\}, \{2\}, \{1,2\} \}$

Ex: $B = \{5, 2, 1\}$

$$\mathcal{P}(B) = \{ \phi, \{1\}, \{2\}, \{5\}, \{5,2\}, \{2,1\}, \{5,1\}, \{5,2,1\} \}$$

Note: $|\mathcal{P}(B)| = 8 = 2^3 = 2^{|B|}$

Theorem: If $S$ is finite, then $|\mathcal{P}(S)| = 2^{|S|}$

**Theorem:** Let $A$ and $B$ be sets. Then, $A = B$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$.

**proof:**

($\Rightarrow$) It's clear that if $A = B$, then $\mathcal{P}(A) = \mathcal{P}(B)$.

($\Leftarrow$) Now we must prove "If $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$".

Suppose $\mathcal{P}(A) = \mathcal{P}(B)$.

To show that $A = B$ we

will show $A \subseteq B$ and $B \subseteq A$.

$\boxed{\text{Claim 1: } A \subseteq B}$

We know $A \subseteq A$.

So, $A \in \mathcal{P}(A)$.

Then, since $\mathcal{P}(A) = \mathcal{P}(B)$, we know $A \in \mathcal{P}(B)$.

Thus, $A \subseteq B$.

$\boxed{\text{claim 2: } B \subseteq A}$

You can do this proof the same way as claim 1, but let's change it up.

Let $b \in B$.

Then, $\{b\} \subseteq B$.

So, $\{b\} \in \mathcal{P}(B)$

Since $\mathcal{P}(B) = \mathcal{P}(A)$ we have $\{b\} \in \mathcal{P}(A)$

Thus, $\{b\} \subseteq A$.

Hence $b \in A$.

So, $B \subseteq A$.

By claim 1 and 2 we know $A = B$.

$\blacksquare$

# HW 2 #3

Let $A, B, C$ be sets.
**Prove:** If $A \subseteq B$,
then $A - C \subseteq B - C$

**proof:** Assume $A \subseteq B$.
Let's prove this implies
that $A - C \subseteq B - C$.

Let $x \in A - C$.
Then, $x \in A$ and $x \notin C$.
Since $x \in A$ and $A \subseteq B$,
we know that $x \in B$.

So, $x \in B$ and $x \notin C$.

Thus, $x \in B - C$.

We have shown that $A - C \subseteq B - C$. $\blacksquare$

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**Q:** Is the converse true? Ie,

If $A - C \subseteq B - C$, then $A \subseteq B$.

**No.**

$A = \{1, 2\}$

$B = \{4, 2\}$

$C = \{1, 4\}$

$A - C = \{2\}$

$B - C = \{2\}$.

So, $A - C \subseteq B - C$, but $A \nsubseteq B$.

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Recall:

Statement: If $P$, then $Q$.

Converse: If $Q$, then $P$.

Contrapositive: If $\neg Q$, then $\neg P$.

Def: When every element of a set $\mathcal{A}$ is itself a set then we call $\mathcal{A}$ a family or collection of sets.

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Ex: $\mathcal{P}(A)$ is a family of sets.

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$\boxed{\text{HW 2}}$ -

(4) Let $A, B$ be sets.
Prove that $A \subseteq B$ iff $A - B = \phi$.

<u>Proof:</u>

($\Rightarrow$) Suppose $A \subseteq B$.
We must show that $A - B = \phi$.
What would happen if $A - B \neq \phi$?
Then there would exist $x \in A - B$.
Then $x \in A$ and $x \notin B$.
But then $A \nsubseteq B$. ← contradicts $A \subseteq B$
So we must have $A - B = \phi$.

($\Leftarrow$) Suppose $A - B = \phi$.

We must show $A \subseteq B$.

Pick some $a \in A$.

We must show $a \in B$.

What if $a \notin B$?

Then, $A - B \neq \emptyset$ because $a \in A - B$.

Contradiction.

Thus, $a \in B$.

Therefore, $A \subseteq B$. $\blacksquare$