(HW 3 TOPIC) — Equivalence relations & Well-defined operations & Modulo n

<u>Def:</u> A <u>relation</u> $\sim$ on a set S is a subset of $S \times S$. If (x,y) is an element of $\sim$ then we write $x \sim y$ and say that <u>x is related to y</u>.
If (x,y) is not an element of $\sim$ then we write $x \not\sim y$ and say that <u>x is not related to y</u>.

Ex: $S = \{\square, \$, ¢\}$

$$\sim = \{(\square, ¢), (\square, \square), (\square, \$), (¢, \square), (¢, \$)\}$$

$\square \sim ¢$ since $(\square, ¢) \in \sim$.

$¢ \sim \square$ since $(¢, \square) \in \sim$.

$\square \sim \square$ since $(\square, \square) \in \sim$.

$\$ \nsim \$$ since $(\$, \$) \notin \sim$.

More common way to define a relation is like the following example.

Ex: $S = \mathbb{Z}$

Define $x \sim y$ to mean $x < y$.
Then $\sim$ is a relation on $\mathbb{Z}$,
we have

$2 \sim 3$ since $2 < 3$.
$2 \nsim -1$ since $2 \nless -1$.

How can we think of $\sim$ as a subset of $\mathbb{Z} \times \mathbb{Z}$?

$$\sim = \{(x,y) \mid x < y\}$$

$$= \{(2,3), (-100,0), (7,777), \ldots\}$$

infinitely many more elements

can think of an equivalence relation as an "equals" sign.

Def: Let $\sim$ be a relation on a set S.

We say that $\sim$ is an <u>equivalence relation</u> on S if

① (reflexive) $x \sim x$ for all $x \in S$.

② (symmetric) If $x, y \in S$ and $x \sim y$, then $y \sim x$.

③ (transitive) If $x, y, z \in S$ and $x \sim y$ and $y \sim z$, then $x \sim z$.

Ex: Let $S = \{-15, 2, 0\}$

and $\sim \; = \{(0,0), (0,2), (2,0), (-15,2)\}$

(reflexive?) $0 \sim 0$ but $2 \not\sim 2$ and $(-15) \not\sim (-15)$.

So, $\sim$ is not reflexive.

(symmetric?) NO! $(-15) \sim 2$ but $2 \not\sim (-15)$.

(transitive?)

$$\left.\begin{array}{l} 0 \sim 0 \text{ and } 0 \sim 2 \implies 0 \sim 2 \; \checkmark \\ 0 \sim 2 \text{ and } 2 \sim 0 \implies 0 \sim 0 \; \checkmark \\ 2 \sim 0 \text{ and } 0 \sim 0 \implies 2 \sim 0 \; \checkmark \end{array}\right\}$$

$0 \sim 0$ and $0 \sim 0 \implies 0 \sim 0$ ✓

$$\boxed{-15 \sim 2 \text{ and } 2 \sim 0 \text{ but } -15 \not\sim 0}$$

So, $\sim$ is not transitive.

$$\boxed{\sim \text{ is NOT an equivalence relation}}$$

Ex: $S = \{1, 2, 3\}$

$\sim = \{(1,1), (2,2), (3,3), (1,3), (3,1)\}$

Is $\sim$ an equivalence relation? YES

reflexive

$1 \sim 1$
$2 \sim 2$
$3 \sim 3$

Yes

symmetric

Yes

If $x \sim y$, then $y \sim x$.

Ex: $1 \sim 3$ and $3 \sim 1$.

transitive

- $1 \sim 1$ and $1 \sim 1$. And $1 \sim 1$.
- $1 \sim 1$ and $1 \sim 3$. And $1 \sim 3$.
- $2 \sim 2$ and $2 \sim 2$. And $2 \sim 2$.
- $3 \sim 3$ and $3 \sim 3$. And $3 \sim 3$.
- $3 \sim 3$ and $3 \sim 1$. And $3 \sim 1$.
- $1 \sim 3$ and $3 \sim 3$. And $1 \sim 3$.
- $1 \sim 3$ and $3 \sim 1$. And $1 \sim 1$.
- $3 \sim 1$ and $1 \sim 1$. And $3 \sim 1$.
- $3 \sim 1$ and $1 \sim 3$. And $3 \sim 3$.

Yes

Def: Let $\sim$ be an equivalence relation on a set S. Let $x \in S$. The *equivalence class* of x is

$$\overline{x} = \{ y \in S \mid x \sim y \}$$

$\overline{x}$ consists of all elements y that are related to x.

It doesn't matter if you define $\overline{x}$ as $\{ y \in S \mid x \sim y \}$ or $\{ y \in S \mid y \sim x \}$ since $\sim$ is symmetric.

Another notation for $\overline{x}$ is $[x]$.

Ex: $S = \{1, 2, 3\}$

$\sim = \{(1,1), (2,2), (3,3), (1,3), (3,1)\}$

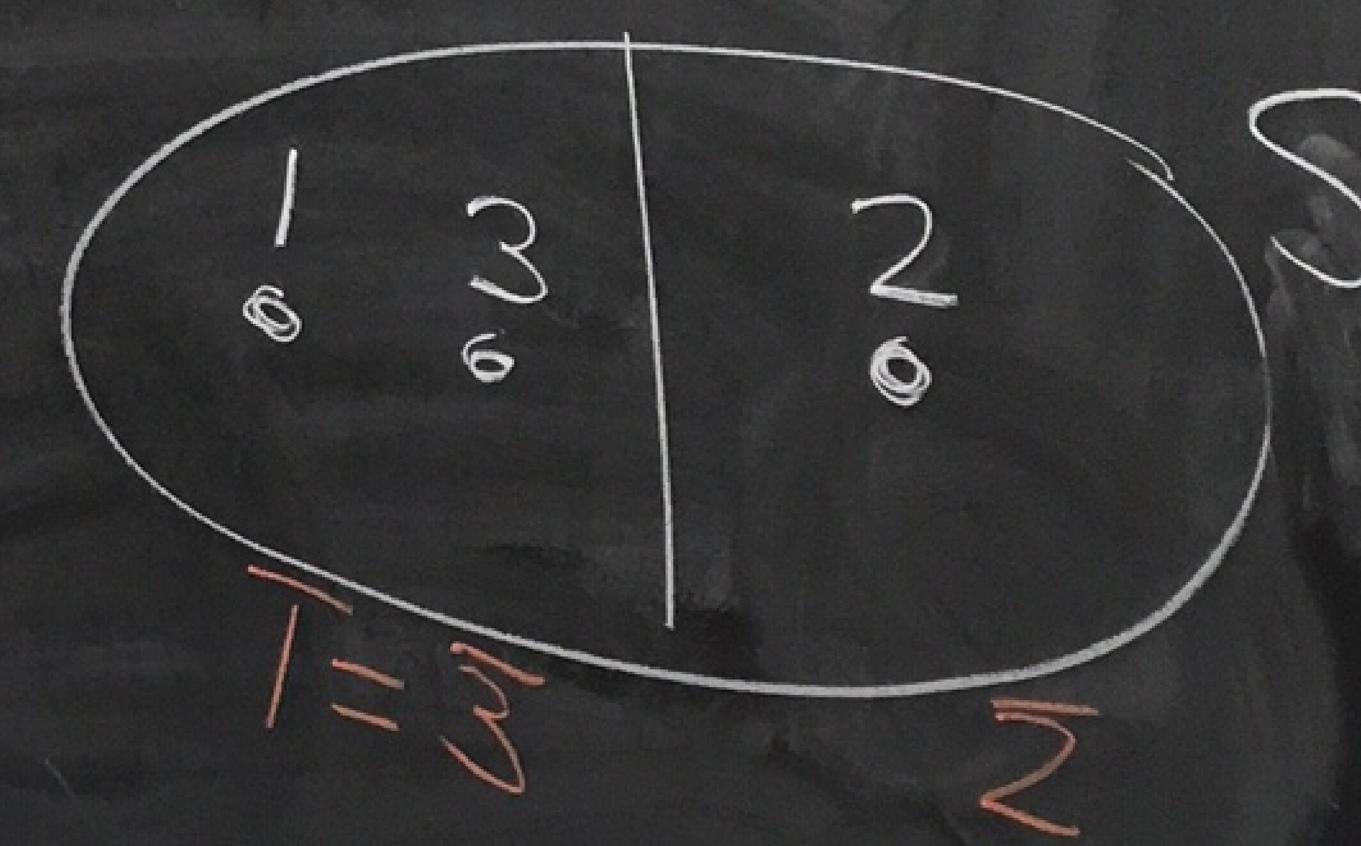
$\sim$ is an equivalence relation.

$\overline{1} = \{y \in S \mid 1 \sim y\} = \{1, 3\}$

$\overline{2} = \{y \in S \mid 2 \sim y\} = \{2\}$

$\overline{3} = \{y \in S \mid 3 \sim y\} = \{1, 3\}$

There are 2 equivalence classes $\overline{1} = \overline{3}$ and $\overline{2}$



Def: Let S be a set and $\sim$ be an equivalence relation on S. We denote the set of equivalence classes as $S/\sim$ read: "S mod $\sim$"

Ex: In the previous example

$$S/\sim = \{\bar{1}, \bar{2}\}$$

$\bar{1} = \bar{3}$

Idea:

since $1 \sim 3$ we make 1 and 3 equal in the set of equivalence classes