Math 3450

3/26/24

Continued from last time...

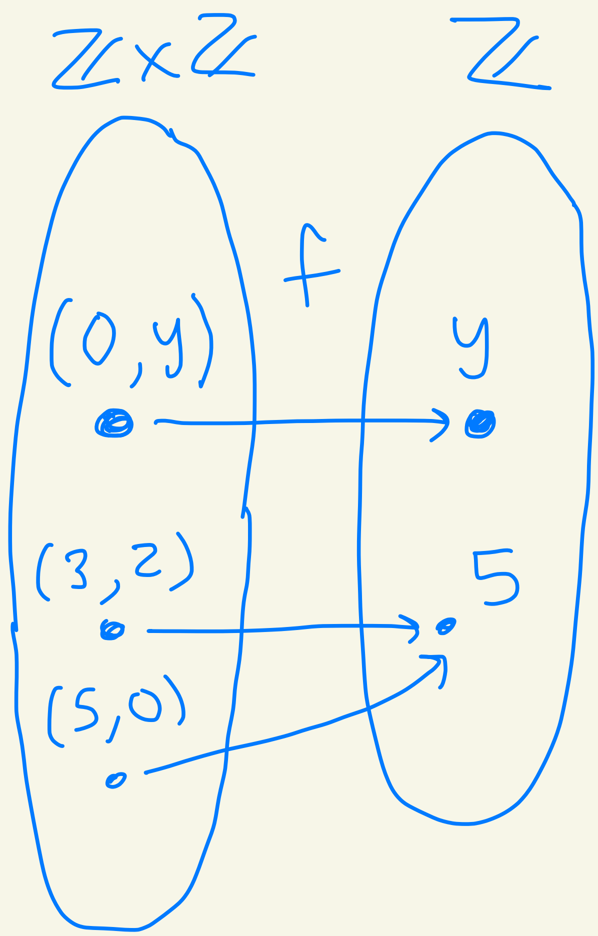
$f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $f(m,n) = m+n$.

**Question:** Is $f$ onto?
Is $f$ 1-1?

**Claim:** $f$ is onto

**proof:**
Let $y \in \mathbb{Z}$.
Then, $(0,y) \in \mathbb{Z} \times \mathbb{Z}$
and $f(0,y) = 0 + y$
$= y$.

$\blacksquare$



**Claim:** $f$ is not 1-1

**proof:** $f(3,2) = 5 = f(5,0)$

but $(3,2) \neq (5,0)$.

See picture above. $\blacksquare$

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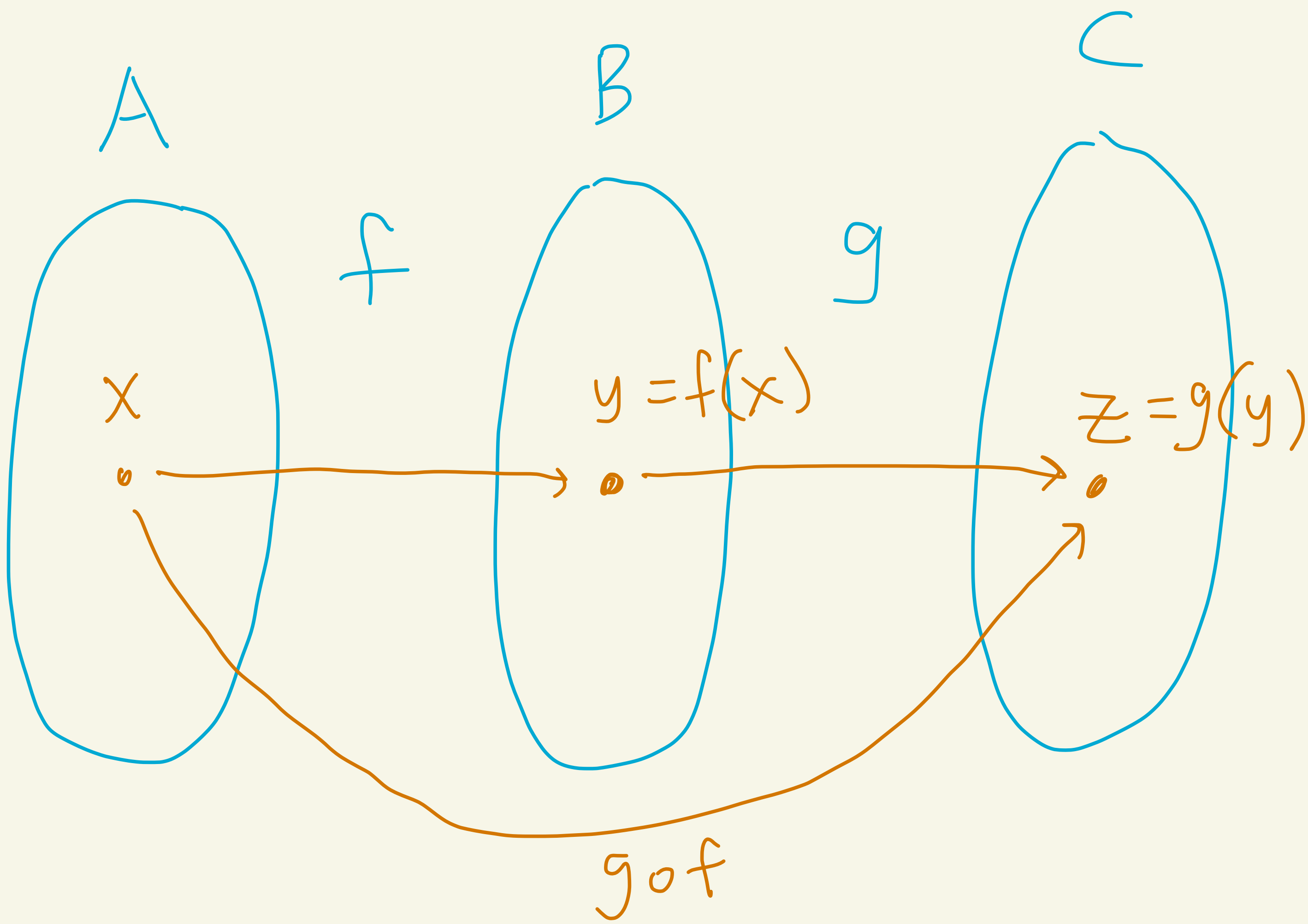
**Theorem:** Let $A, B, C$ be sets and $f : A \to B$ and $g : B \to C$.

1. If $f$ and $g$ are both onto, then $g \circ f$ is onto.
2. If $f$ and $g$ are both 1-1, then $g \circ f$ is 1-1.

③ If $f$ and $g$ are both bijections (1-1 and onto), then $g \circ f$ is a bijection.

proof:

① Suppose $f$ and $g$ are both onto.
Note $g \circ f : A \to C$.



Let $z \in C$.

Since $g$ is onto $C$, there exists $y \in B$ where $g(y) = z$.

Since $f$ is onto $B$, there exists $x \in A$ where $f(x) = y$.

Then $(g \circ f)(x) = g(f(x))$
$= g(y) = z$.

So, $g \circ f$ is onto [because there exists $x \in A$ with $(g \circ f)(x) = z$]

(2) Suppose $f$ and $g$ are both 1-1.

Suppose $(g \circ f)(x_1) = (g \circ f)(x_2)$
where $x_1, x_2 \in A$.

Then, $g(f(x_1)) = g(f(x_2))$.

Since $g$ is 1-1 and
$g(f(x_1)) = g(f(x_2))$,
this implies that $f(x_1) = f(x_2)$.

Since $f$ is 1-1 and
$f(x_1) = f(x_2)$ this
implies that $x_1 = x_2$.

So, $(g \circ f)(x_1) = (g \circ f)(x_2)$.
implies that $x_1 = x_2$.

Thus, $g \circ f$ is 1-1.

(3) Suppose $f$ and $g$ are both bijections (1-1 and onto).

By 1, this implies that
$g \circ f$ will be onto.

By 2, this implies that
$g \circ f$ will be 1-1.

So, $g \circ f$ is a bijection. ∎

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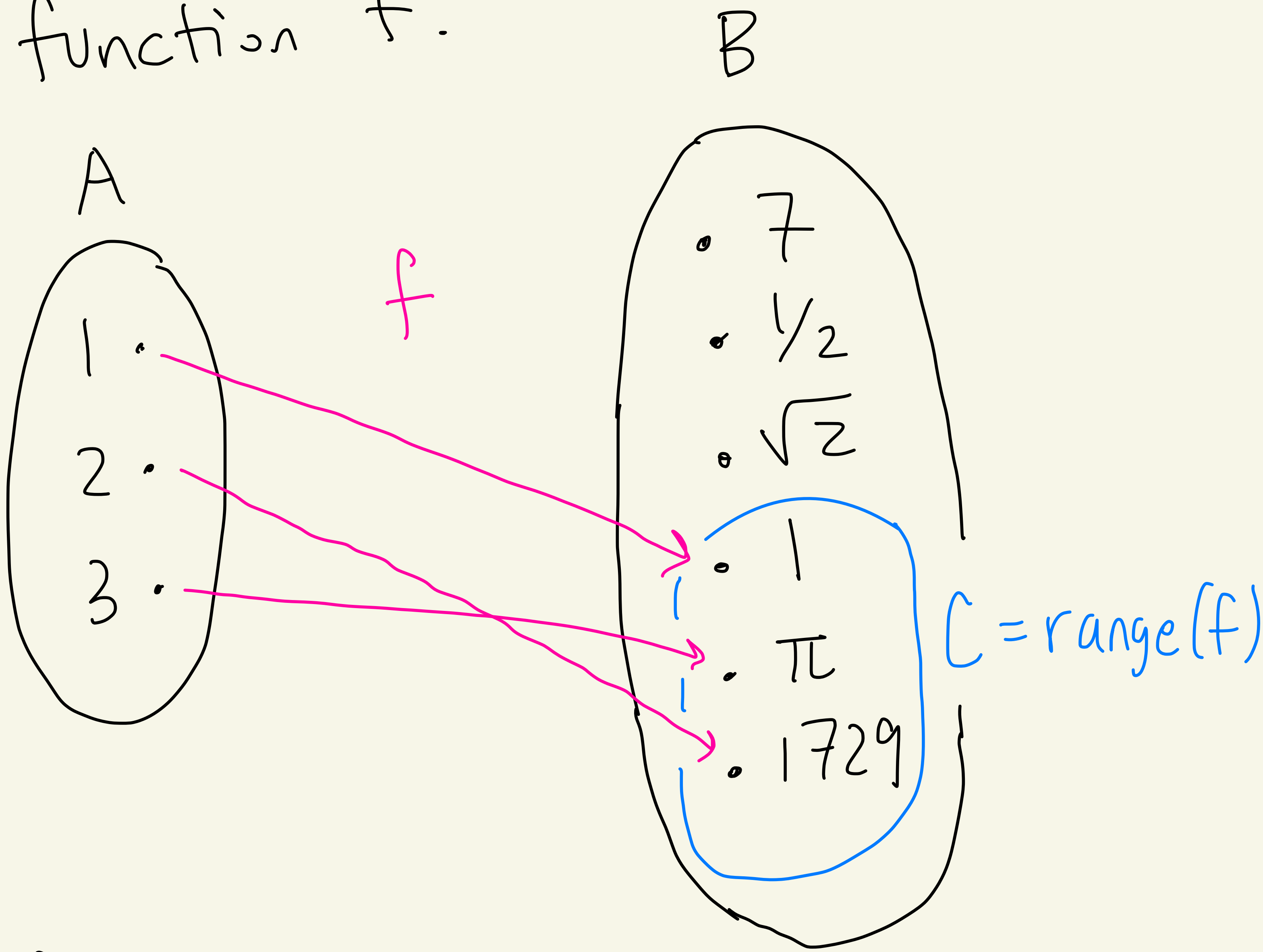
Now we talk about inverse functions.

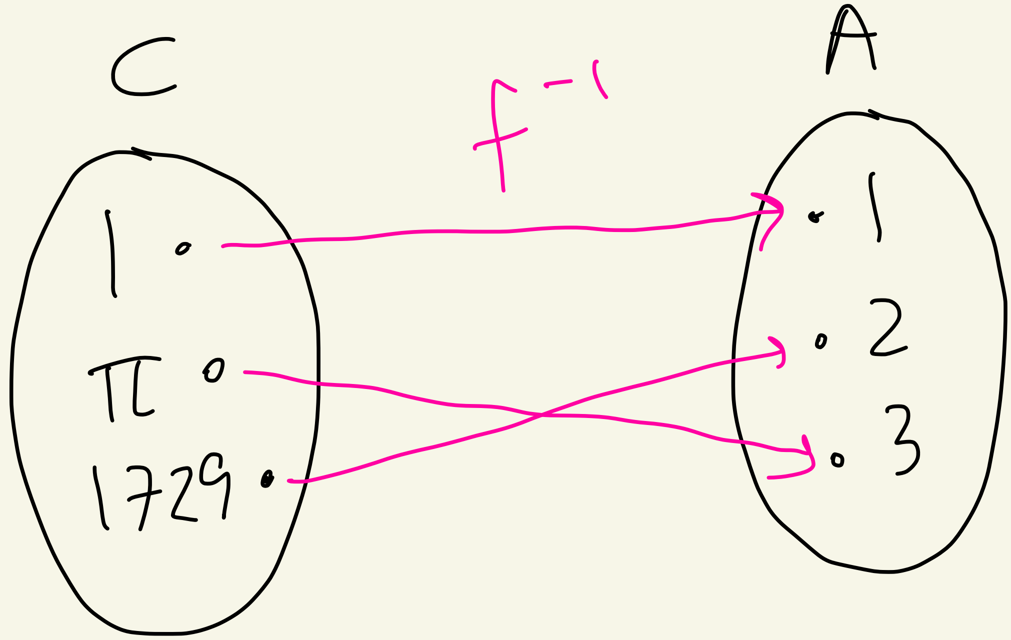
Ex: Consider the following function $f$.




$f$ is one-to-one.

So we can create $f^{-1}: C \to A$ by reversing the arrows

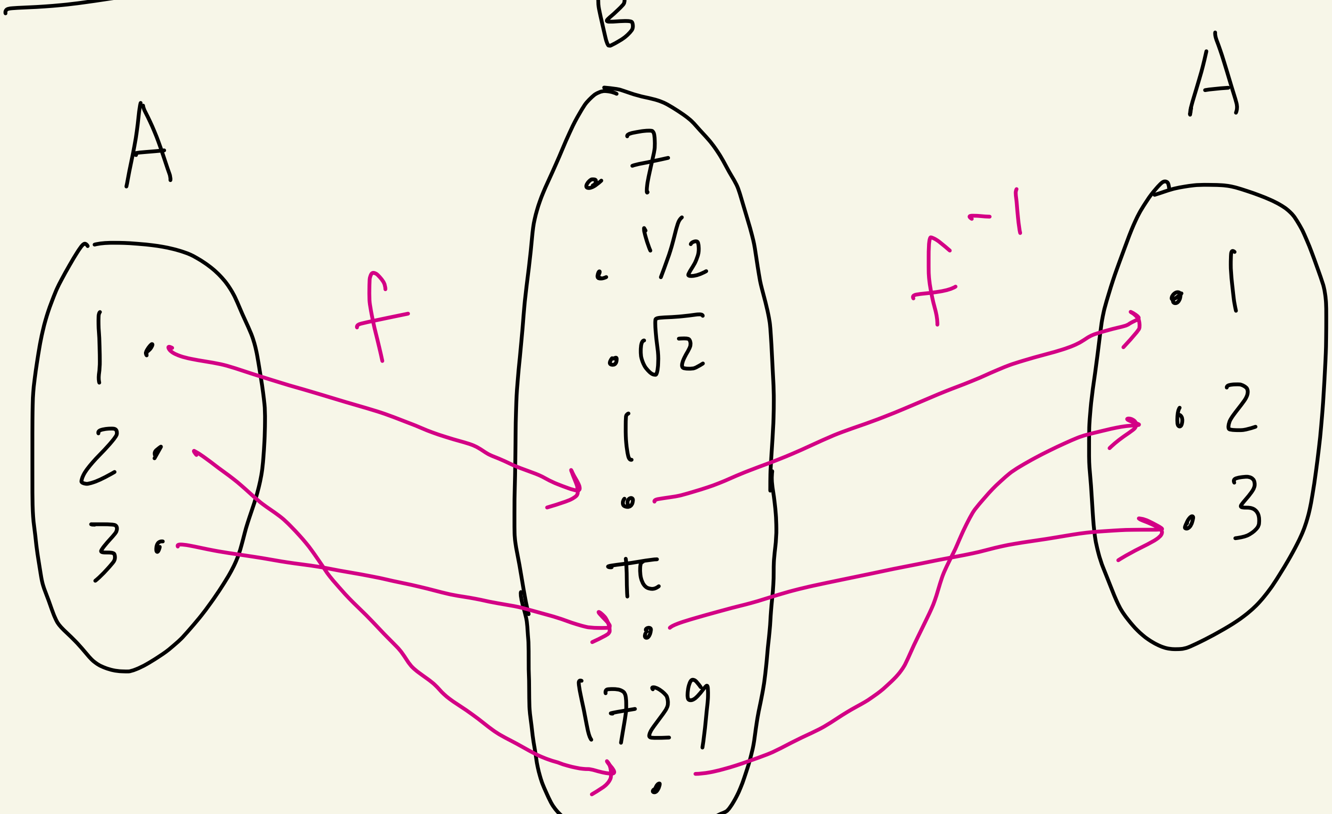
$f^{-1}$ will be well-defined since $f$ is 1-1 and so each element of $C$ only has one arrow to reverse.

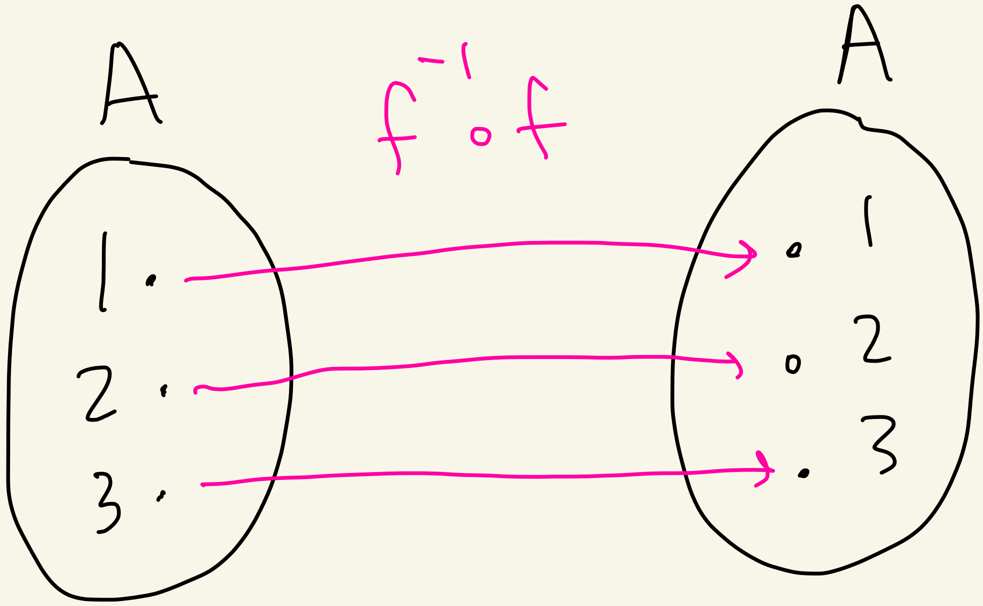



$\mathrm{domain}(f^{-1}) = C = \mathrm{range}(f)$

$\mathrm{range}(f^{-1}) = A = \mathrm{domain}(f)$

What is $f^{-1} \circ f$ ?





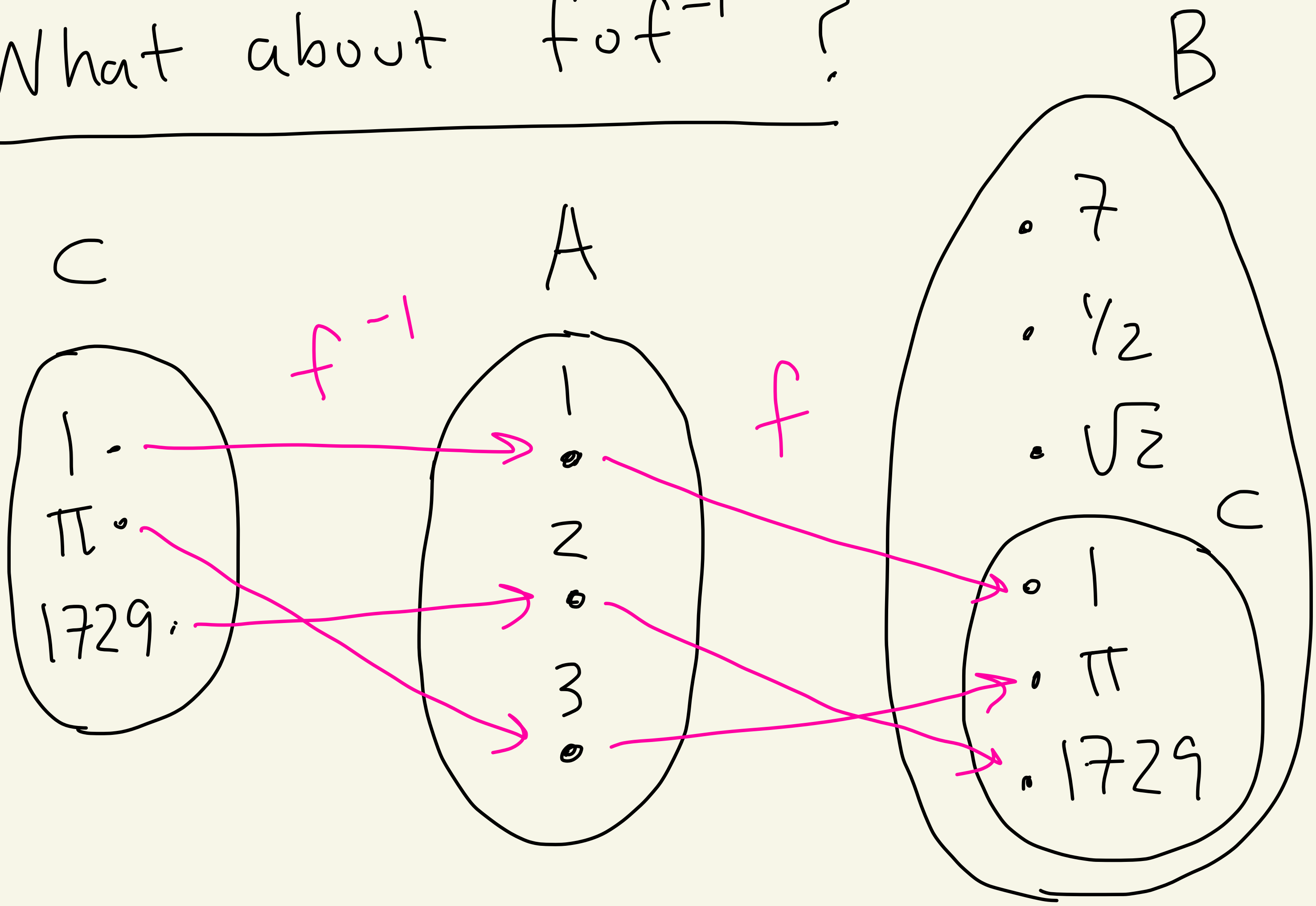

$$(f^{-1} \circ f)(1) = f^{-1}(f(1)) = f^{-1}(1) = 1$$

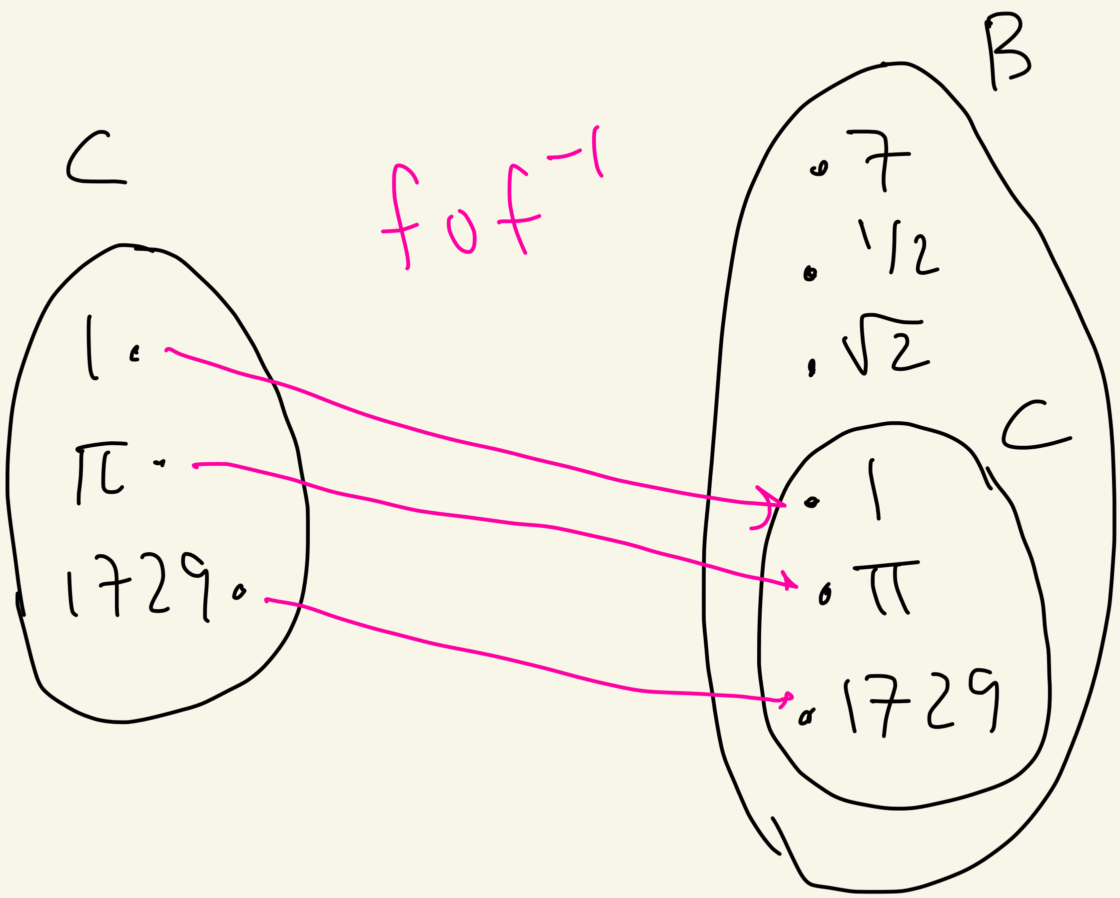
$$(f^{-1} \circ f)(2) = f^{-1}(f(2)) = f^{-1}(1729) = 2$$

$$(f^{-1} \circ f)(3) = f^{-1}(f(3)) = f^{-1}(\pi) = 3$$

Thus, $f^{-1} \circ f = i_A$ (the identity function on $A$)

# What about $f \circ f^{-1}$?



We see that

$$(f \circ f^{-1})(z) = z = i_C(z)$$

for all $z \in C$.

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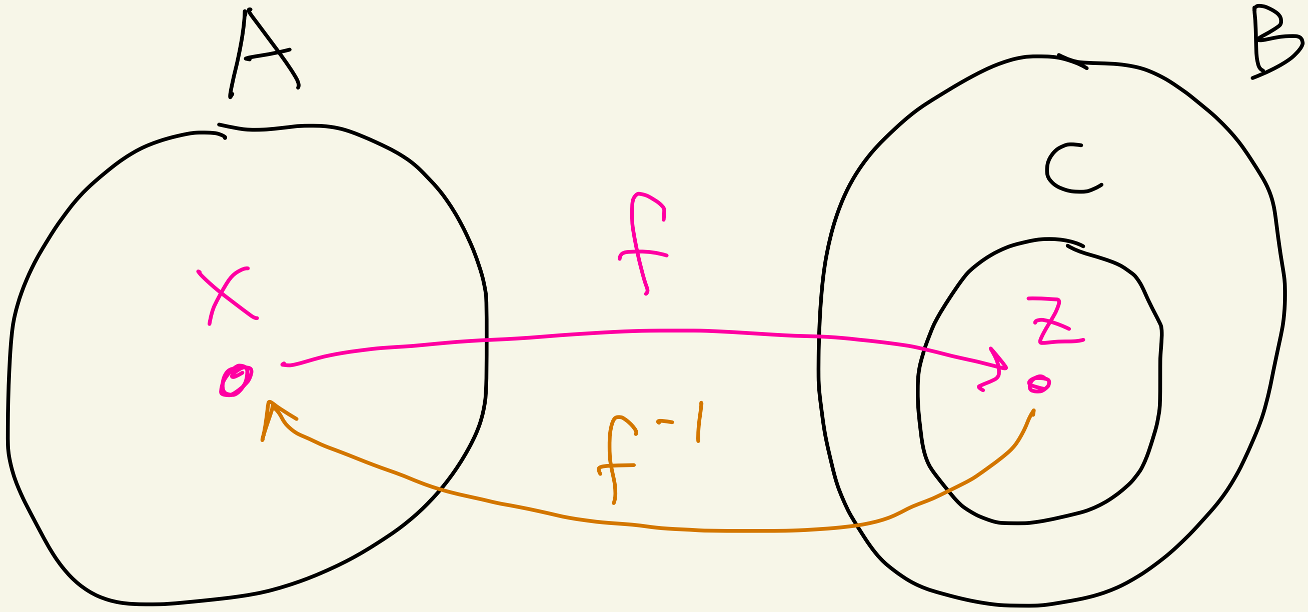
**Def:** Let $A$ and $B$ be sets. Let $f: A \to B$ be a one-to-one function. Let $C = \text{range}(f)$. Define the *inverse function of $f$* to be $f^{-1}: C \to A$ such that $f^{-1}(z) = x$ where $f(x) = z$.




**Note:** $f^{-1}$ is well-defined because $f$ is one-to-one. There is one and only one arrow to reverse for each $z$ in $C$.