Math 3450
4/16/24

Recall:

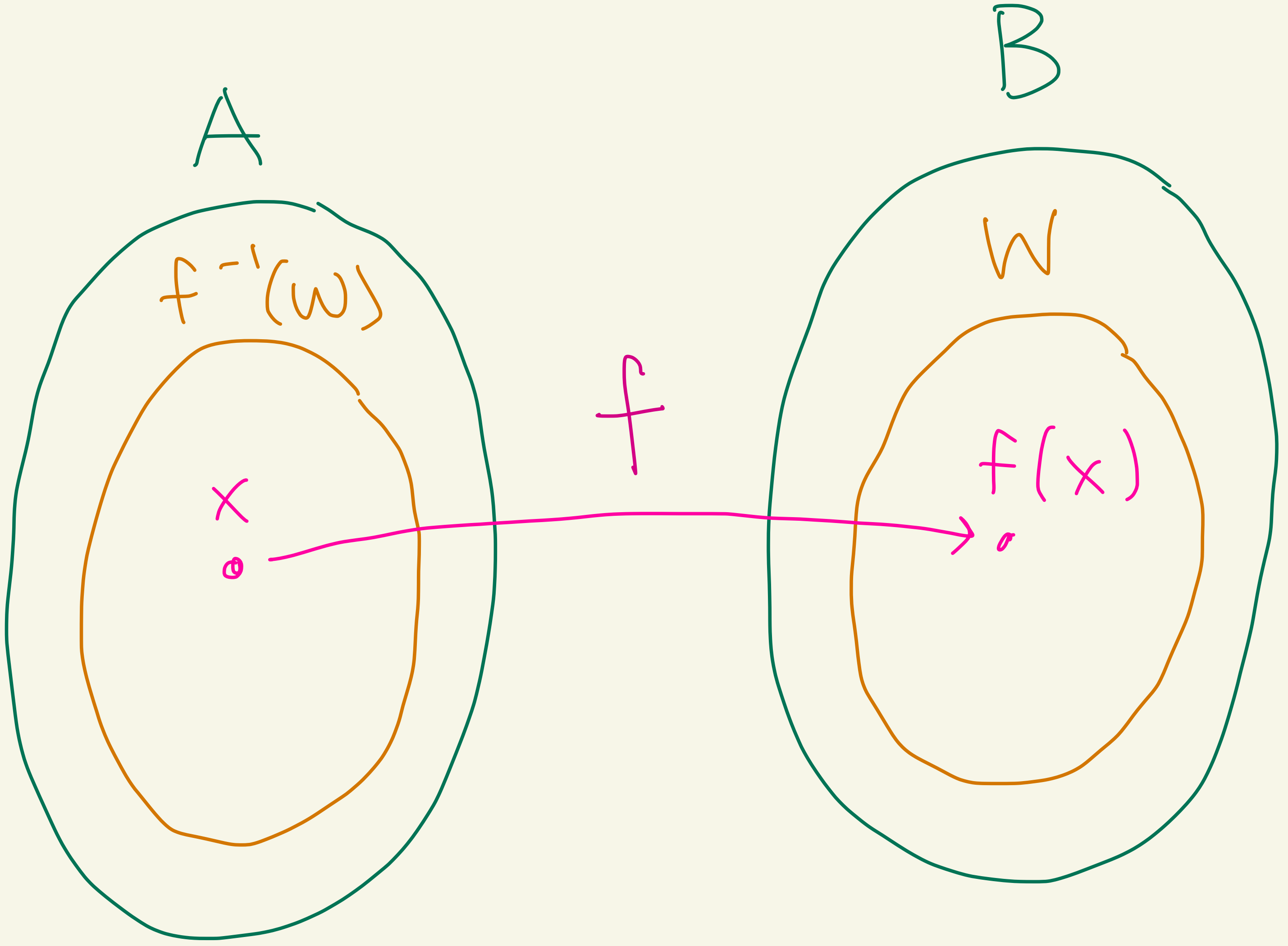
$f: A \to B$

$W \subseteq B$

$$f^{-1}(W) = \{ x \in A \mid f(x) \in W \}$$

Key:

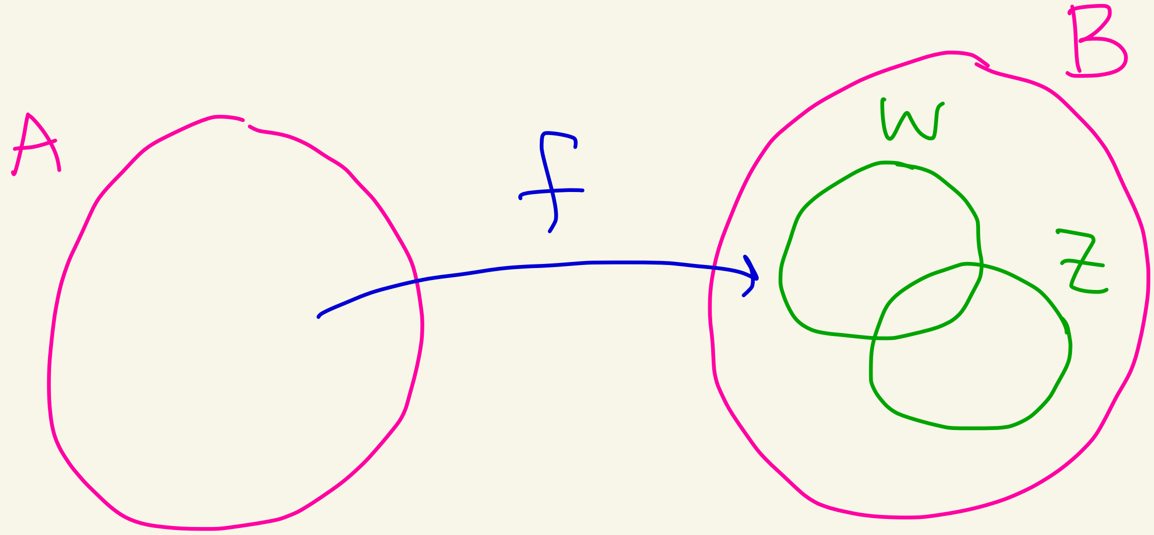
$x \in f^{-1}(W)$ means $f(x) \in W$



<u>Theorem:</u> Let A, B be sets.

Let $f: A \to B$.

Let $W \subseteq B$ and $Z \subseteq B$.



Then:

① $f^{-1}(W \cap Z) = f^{-1}(W) \cap f^{-1}(Z)$

② $f^{-1}(W \cup Z) = f^{-1}(W) \cup f^{-1}(Z)$

③ $A - f^{-1}(W) = f^{-1}(B - W)$

④ If $W \subseteq Z$, then $f^{-1}(W) \subseteq f^{-1}(Z)$

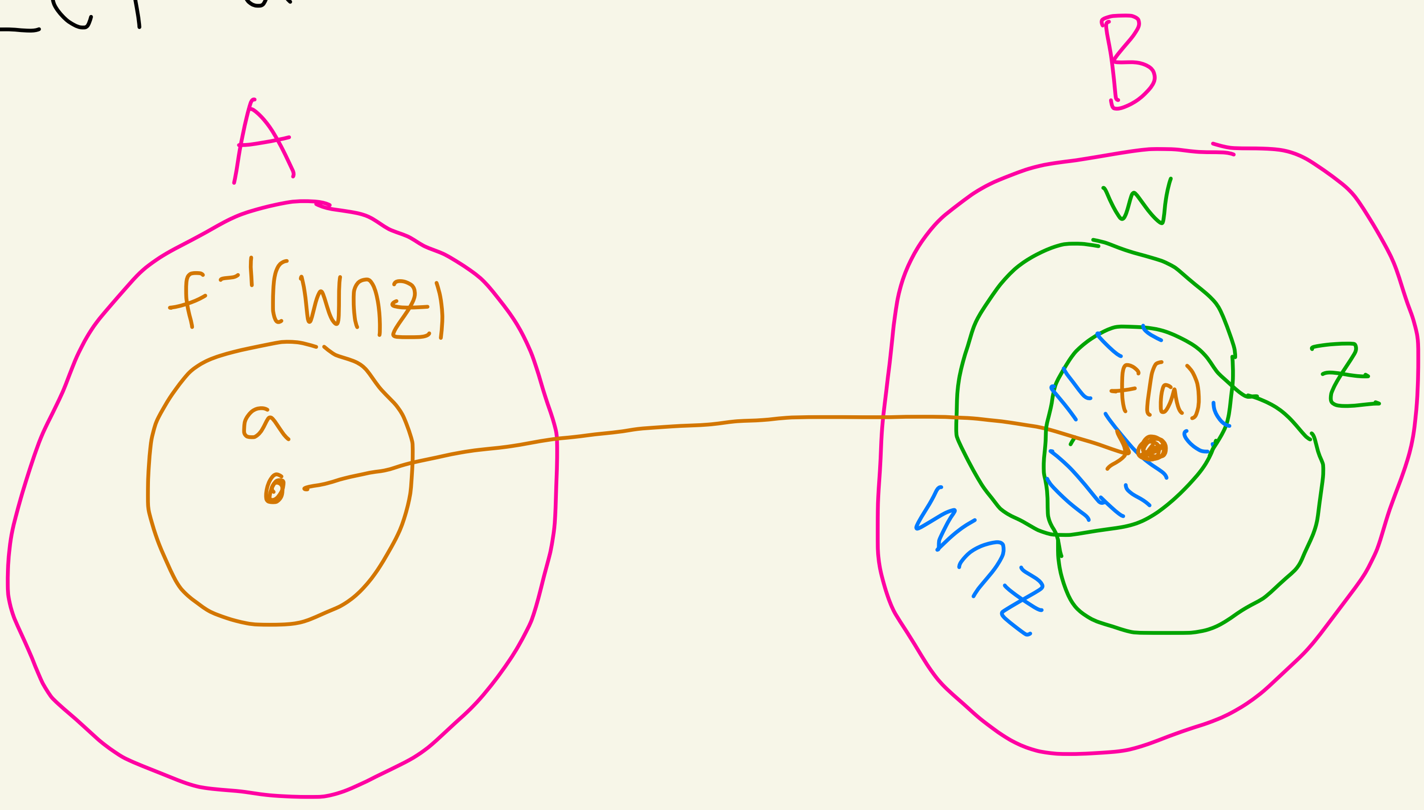
proof:

① Let's show that

$f^{-1}(W \cap Z) = f^{-1}(W) \cap f^{-1}(Z)$.

$\boxed{\subseteq}$:

Let $a \in f^{-1}(W \cap Z)$.



Then, $f(a) \in W \cap Z$.

So, $f(a) \in W$ and $f(a) \in Z$.

Thus, $a \in f^{-1}(W)$ and $a \in f^{-1}(Z)$.

Therefore, $a \in f^{-1}(W) \cap f^{-1}(Z)$.

$\boxed{\supseteq}$:

Let $x \in f^{-1}(W) \cap f^{-1}(Z)$.

Then, $x \in f^{-1}(W)$ and $x \in f^{-1}(Z)$.

So, $f(x) \in W$ and $f(x) \in Z$

Thus, $f(x) \in W \cap Z$.

So, $x \in f^{-1}(W \cap Z)$.

By $(\subseteq)$ and $(\supseteq)$ we get

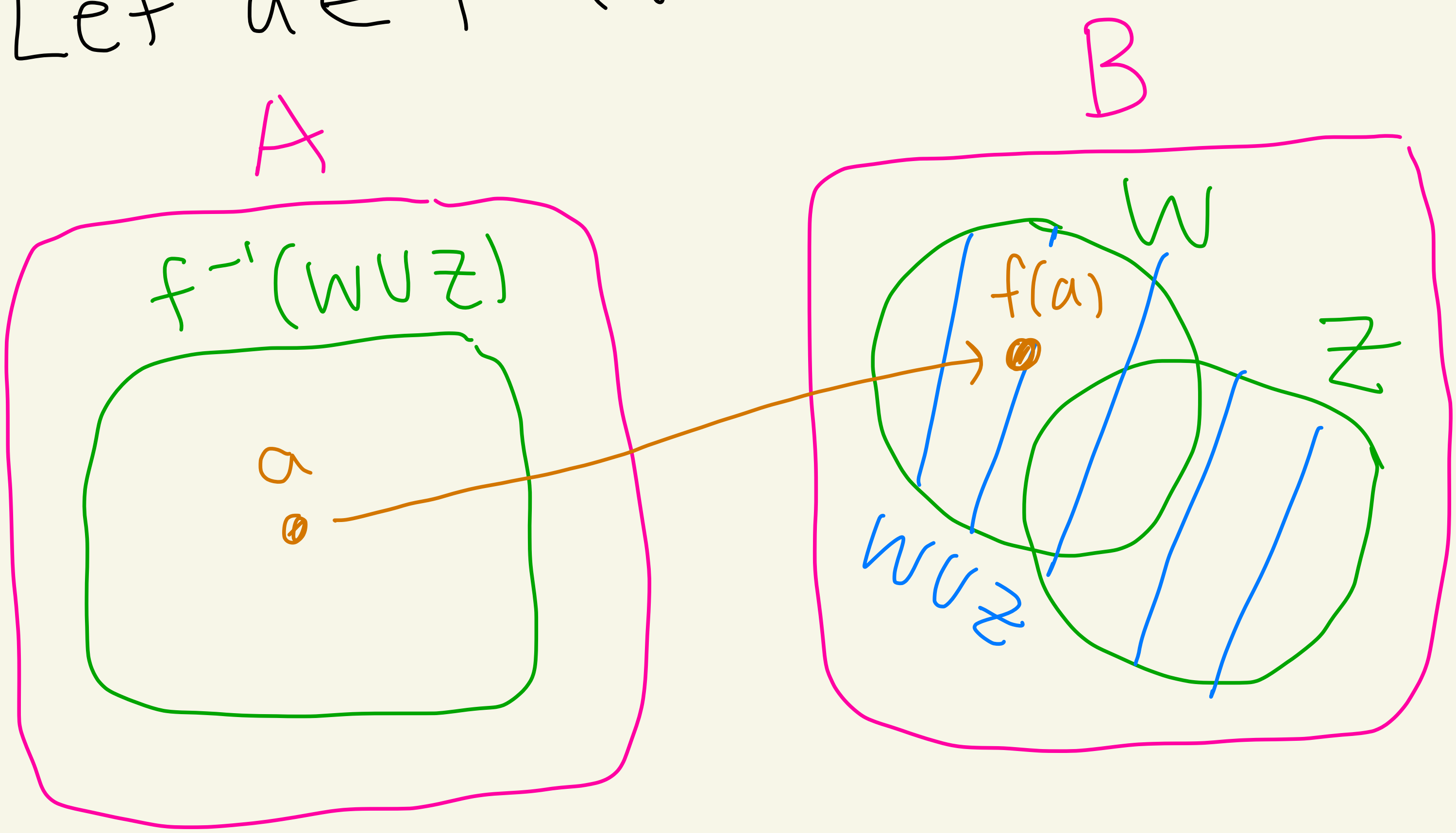
$f^{-1}(W \cap Z) = f^{-1}(W) \cap f^{-1}(Z)$.

② Let's show that

$$f^{-1}(W \cup Z) = f^{-1}(W) \cup f^{-1}(Z)$$

$\boxed{\subseteq}$:

Let $a \in f^{-1}(W \cup Z)$.



So, $f(a) \in W \cup Z$.

Thus, $f(a) \in W$ or $f(a) \in Z$.

Hence, $a \in f^{-1}(W)$ or $a \in f^{-1}(Z)$.

Ergo, $a \in f^{-1}(W) \cup f^{-1}(Z)$.

Thus, $f^{-1}(W \cup Z) \subseteq f^{-1}(W) \cup f^{-1}(Z)$.

$\boxed{\supseteq}$:

Let $x \in f^{-1}(W) \cup f^{-1}(Z)$.

Then, $x \in f^{-1}(W)$ or $f^{-1}(Z)$.

So, $f(x) \in W$ or $f(x) \in Z$.

Hence, $f(x) \in W \cup Z$.

Thus, $x \in f^{-1}(W \cup Z)$.

Hence, $f^{-1}(W) \cup f^{-1}(Z) \subseteq f^{-1}(W \cup Z)$

By ($\subseteq$) and ($\supseteq$) we have that $f^{-1}(W \cup Z) = f^{-1}(W) \cup f^{-1}(Z)$.

Iff version of 2:

$a \in f^{-1}(W \cup Z)$

iff $f(a) \in W \cup Z$

iff $f(a) \in W$ or $f(a) \in Z$

iff $a \in f^{-1}(W)$ or $a \in f^{-1}(Z)$

iff $a \in f^{-1}(W) \cup f^{-1}(Z)$

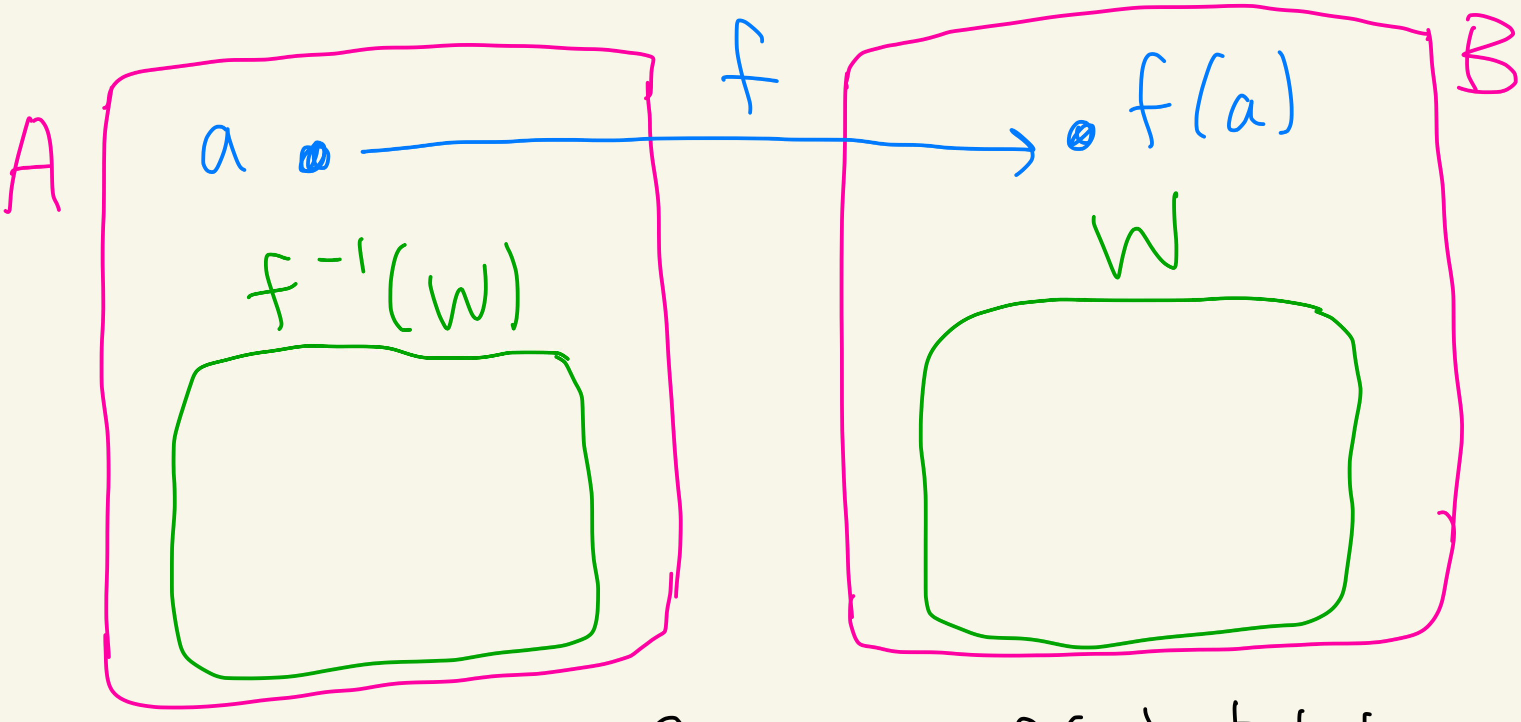
Thus, $f^{-1}(W \cup Z) = f^{-1}(W) \cup f^{-1}(Z)$

(3) Let's show that

$$A - f^{-1}(W) = f^{-1}(B - W)$$

We have that $a \in A - f^{-1}(W)$

iff $a \in A$ and $a \notin f^{-1}(W)$



iff $f(a) \in B$ and $f(a) \notin W$.

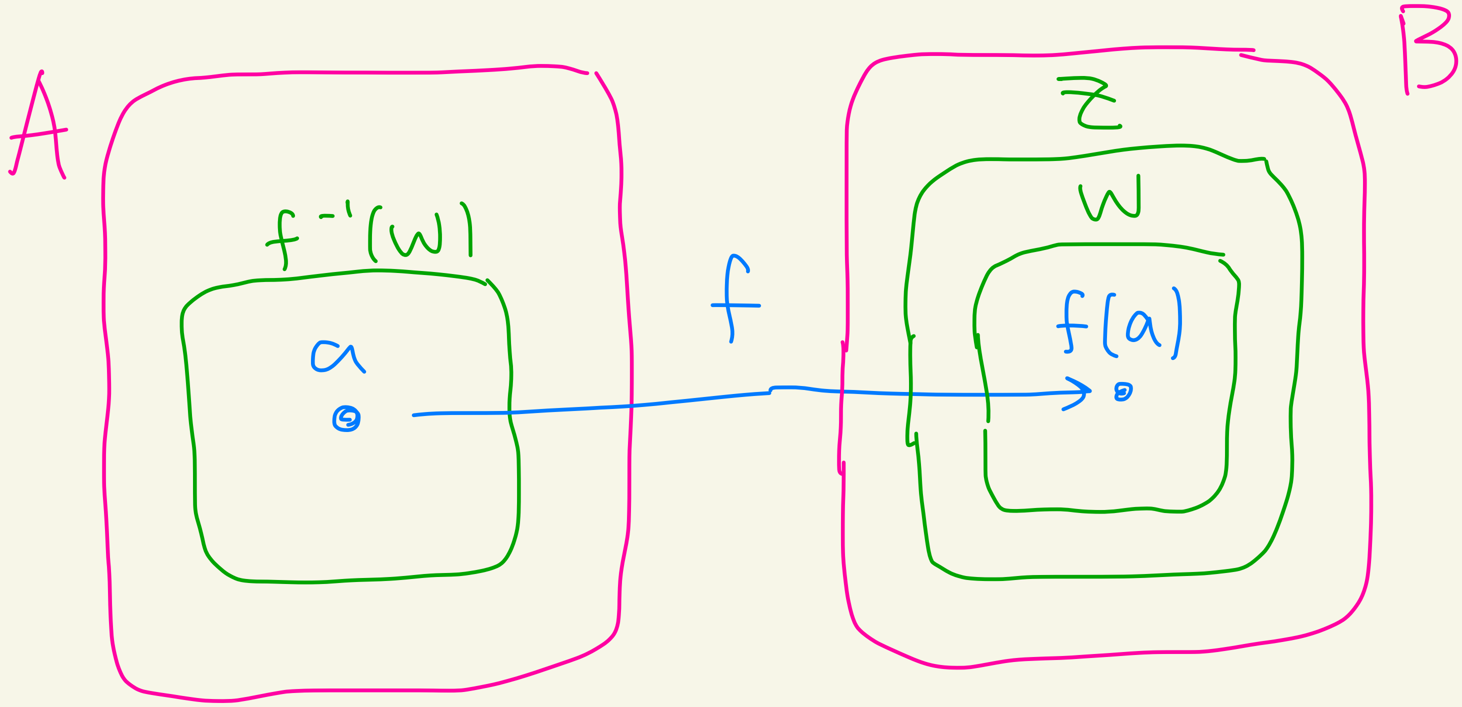
iff $f(a) \in B - W$.

Thus, $A - f^{-1}(W) = f^{-1}(B - W)$.

④ Suppose that $W \subseteq Z$.

Let's prove that $f^{-1}(W) \subseteq f^{-1}(Z)$.

Let $a \in f^{-1}(W)$.



Then, $f(a) \in W$.

Since $f(a) \in W$ and $W \subseteq Z$, we know that $f(a) \in Z$.

Thus, $a \in f^{-1}(Z)$.

Hence, $f^{-1}(W) \subseteq f^{-1}(Z)$.