Math 3450
5/7/24

Practice Test

③ $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$

$g(m, n) = (2m+1, n)$

(c & d)

Show that g is 1-1, but not onto.

(c) Let's show g is 1-1.

Suppose $g(m,n) = g(a,b)$.

Then $(2m+1, n) = (2a+1, b)$.

So, $2m+1 = 2a+1$ and $\boxed{n = b.}$

Solving $2m+1 = 2a+1$ $\quad -1$

gives $2m = 2a$ $\quad$ divide by 2

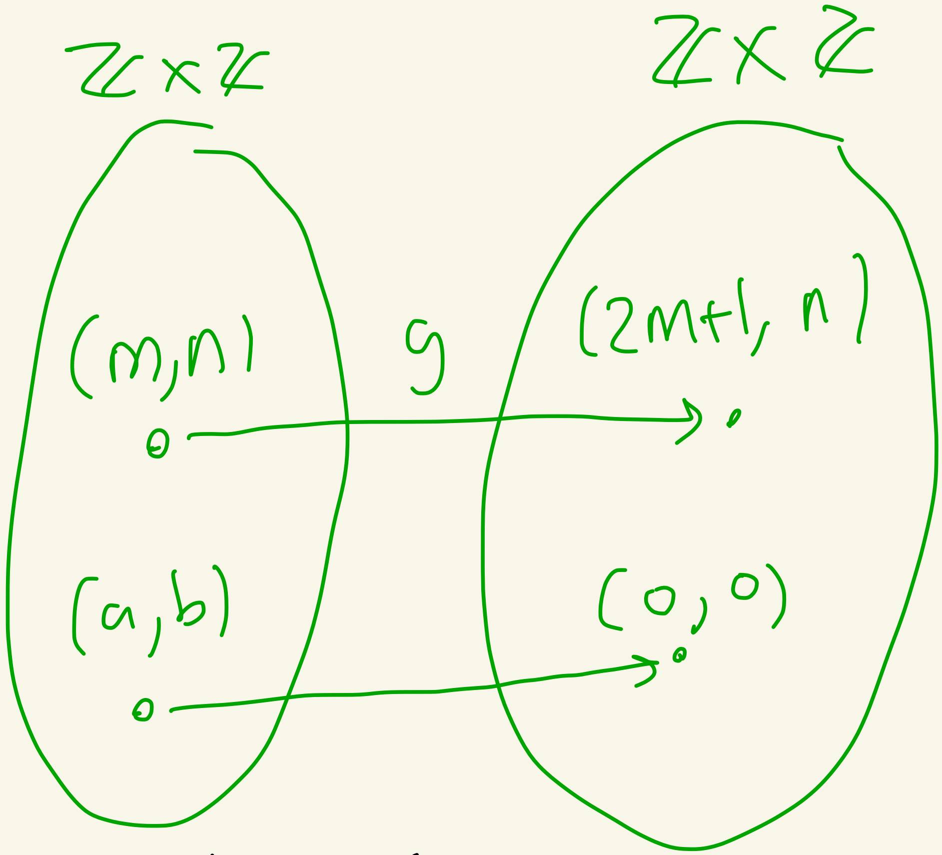
so $\boxed{m = a}$

So, $(m,n) = (a,b)$.

(d) Show g is not onto.

Let's show $(0,0)$ is not in the range of g.

Suppose it was.



Then there would exist $(a,b) \in \mathbb{Z}\times\mathbb{Z}$ with $g(a,b) = (0,0)$

Then $(2a+1, b) = (0,0)$

So, $2a+1 = 0$.

Then $a = -\frac{1}{2} \notin \mathbb{Z}$.

This is impossible!

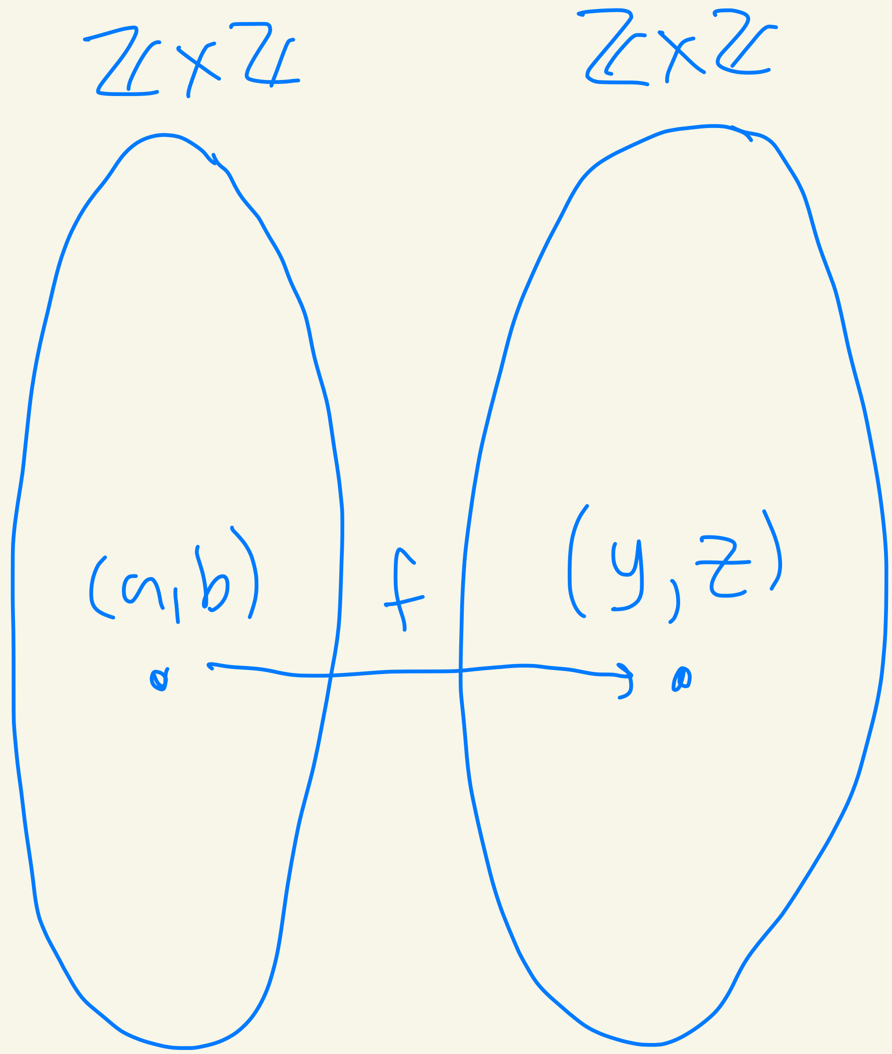
So, $(0,0) \notin \text{range}(g)$ and g is not onto.

New example

Suppose $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$
where $f(m,n) = (m+1, n-3)$.

Show that f is onto.

proof:



Let $(y,z) \in \mathbb{Z} \times \mathbb{Z}$.

We need to find
$(a,b) \in \mathbb{Z} \times \mathbb{Z}$
where
$f(a,b) = (y,z)$.

Need to solve
$(a+1, b-3) = (y,z)$

Need to solve
$$a+1 = y$$
$$b-3 = z$$

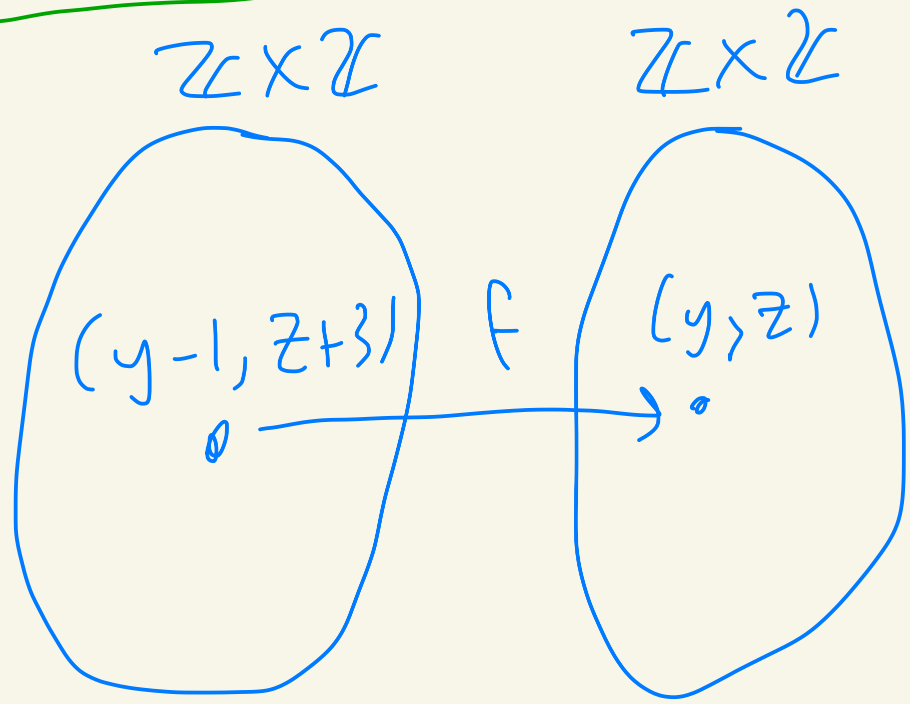
$$\boxed{f(m,n) = (m+1,\, n-3)}$$

We get
$$a = y - 1$$
$$b = z + 3$$

Then
$$f(y-1,\, z+3) = ((y-1)+1,\, (z+3)-3)$$
$$= (y, z).$$

So, f is onto.

$\blacksquare$



f is also 1-1

pf: Suppose $f(a,b) = f(m,n)$

Then $(a+1,\, b-3) = (m+1,\, n-3)$

So, $a+1 = m+1$
$b-3 = n-3$

Thus $a = m$ and $b = n$

So, $(a,b) = (m,n)$. $\blacksquare$

Practice Test

(5) B) $f: A \to B$, $g: B \to C$

(i) If f, g are both onto, then $g \circ f$ is onto.

(ii) If f, g are both 1-1, then $g \circ f$ is 1-1.

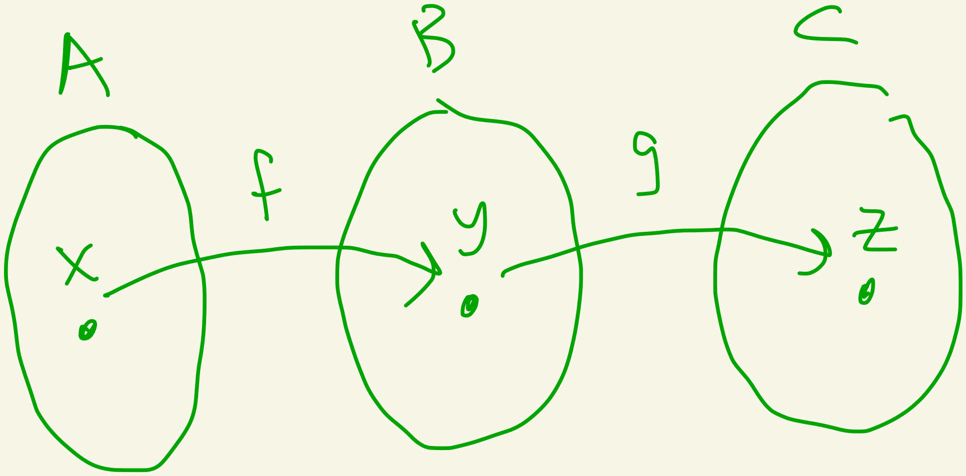
Proof:

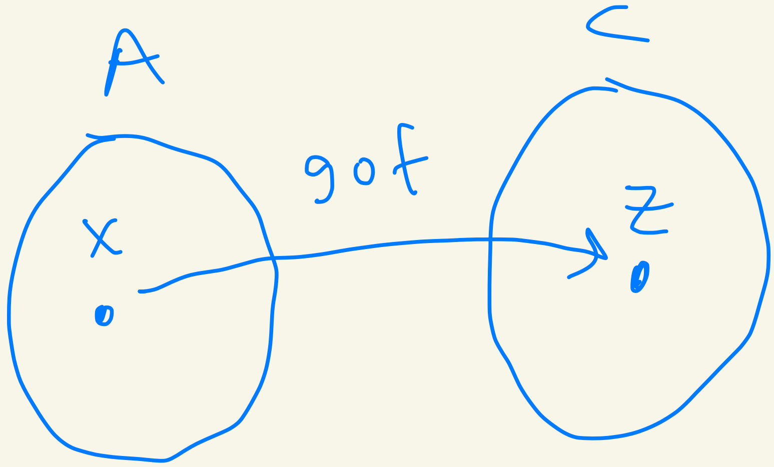
(i)

Let $z \in C$.

Since g is onto $\exists\, y \in B$ with $g(y) = z$.

Since f is onto $\exists\, x \in A$ with $f(x) = y$.





Then, $(g \circ f)(x) = g(f(x)) = g(y) = z$.

So, $g \circ f$ is onto.

(ii) Suppose $(g \circ f)(d) = (g \circ f)(e)$.

Then $g(\boxed{f(d)}) = g(\boxed{f(e)})$.

Since g is 1-1 we know $\boxed{f(d) = f(e)}$.

Since f is 1-1 we know $d = e$.

So, $g \circ f$ is 1-1. $\blacksquare$

Ex: Let $g: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ be $g(m,n) = (2m+1, n)$.

Let $A = \{(0,0), (1,2), (-1,5)\}$.

Find $g^{-1}(A)$.

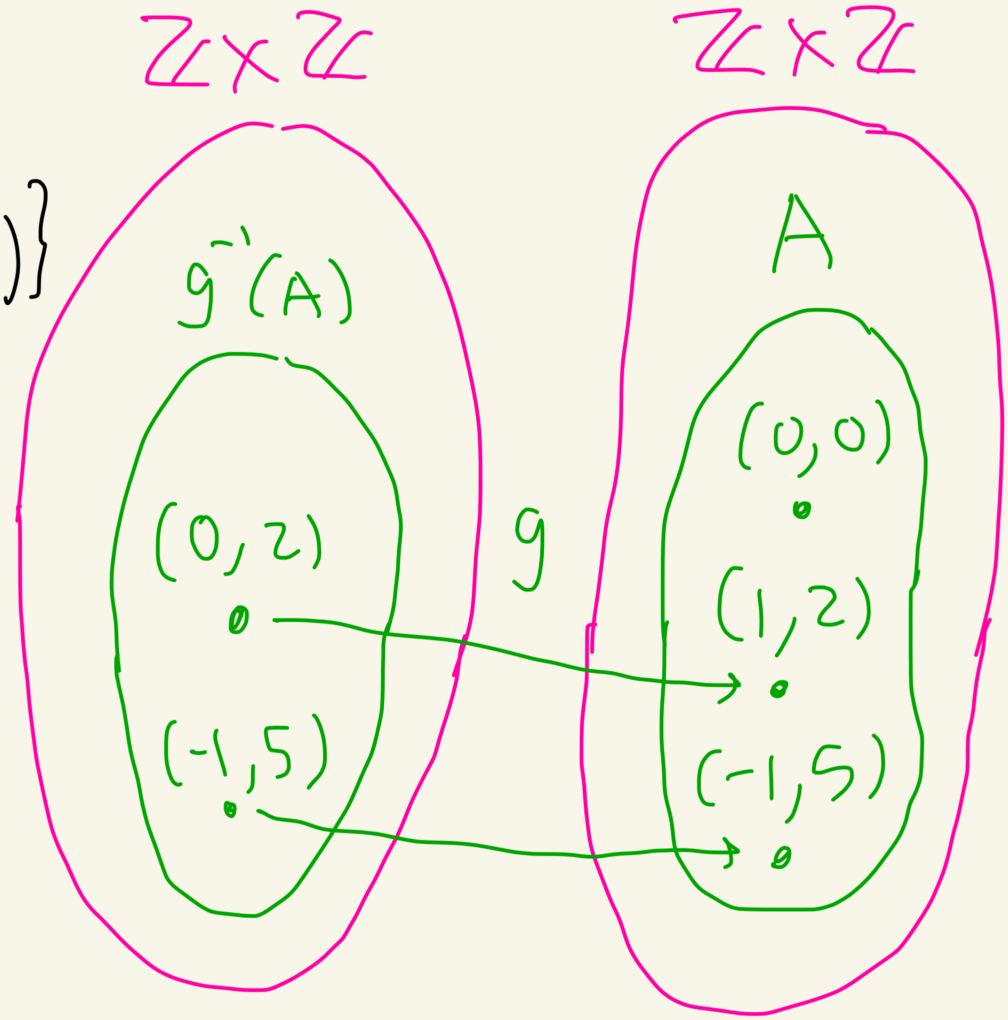
$g^{-1}(A) = \{(0,2), (-1,5)\}$

solve:

$g(m,n) = (1,2)$

$(2m+1, n) = (1,2)$

$m = 0, n = 2$



Test 2

(4) B) $S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$

$(a,b) \sim (c,d)$ means $ad = bc$

$\overline{(a,b)} \odot \overline{(c,d)} = \overline{(ac, bd)}$

show well-defined

proof:

- Let $(a,b), (c,d) \in S$.
  Then $a,b,c,d \in \mathbb{Z}$ and $b \neq 0, d \neq 0$.
  Then, $ac, bd \in \mathbb{Z}$ and $bd \neq 0$.
  So, $\overline{(ac, bd)}$ is a valid equivalence class.

- Suppose $\overline{(a,b)} = \overline{(x,y)}$
  and $\overline{(c,d)} = \overline{(w,z)}$

Need to show that

$$\overline{(a,b)} \odot \overline{(c,d)} = \overline{(ac,bd)}$$

$$\overline{(x,y)} \odot \overline{(w,z)} = \overline{(xw,yz)}$$

are equal.

$(m,n) \sim (q,p)$

$mp = nq$

Need to show

$$\overline{(ac,bd)} = \overline{(xw,yz)}.$$

Need to show $acyz = bdxw$.

We have

$cz = dw$ since $\overline{(c,d)} = \overline{(w,z)}$

$$acyz = bxcz = bxdw = bdxw$$

$ay = bx$ since $\overline{(a,b)} = \overline{(x,y)}$

So, $\overline{(ac,bd)} = \overline{(xw,yz)}$

$\blacksquare$

5) D)

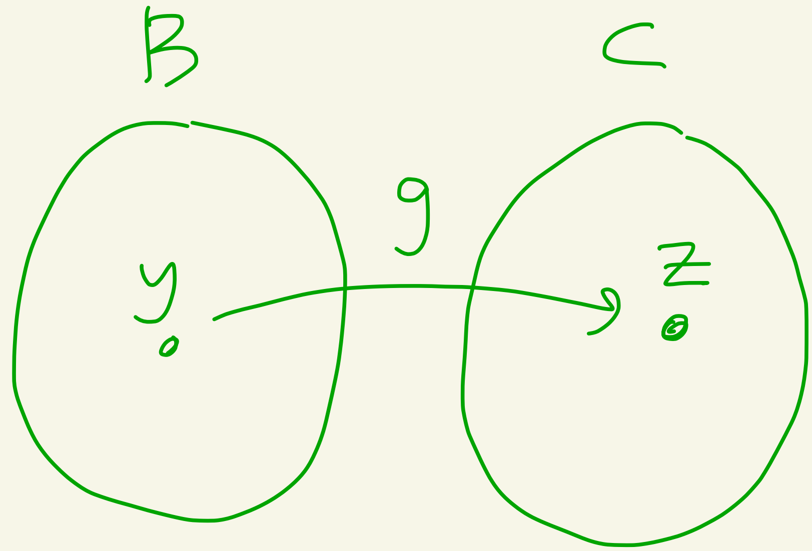
$f: A \to B$, $g: B \to C$

Prove if $g \circ f$ is onto, then g is onto.

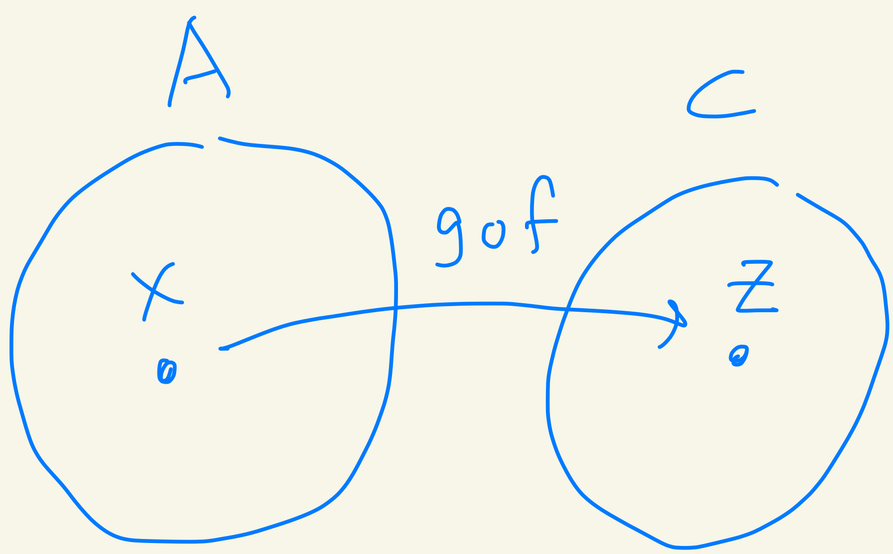
<u>proof:</u>

Let $z \in C$.

We must find $y \in B$ where $g(y) = z$.



Since $g \circ f$ is onto, there exists $x \in A$ with $(g \circ f)(x) = z$.



Let $y = f(x)$.

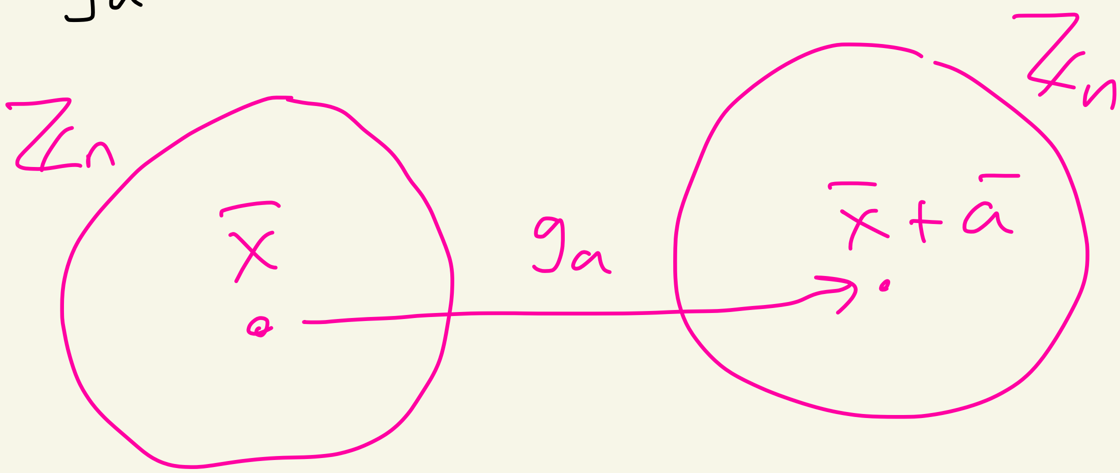
Then, $g(y) = g(f(x)) = z$.

So, g is onto. $\blacksquare$

④ A) $n \in \mathbb{Z}$, $n \geq 2$, $a \in \mathbb{Z}$

$g_a : \mathbb{Z}_n \to \mathbb{Z}_n$, $g_a(\overline{x}) = \overline{x} + \overline{a}$

Show g_a is a bijection.



<u>proof:</u>

(1-1) Suppose $g_a(\overline{x_1}) = g_a(\overline{x_2})$.

Then, $\overline{x_1} + \overline{a} = \overline{x_2} + \overline{a}$.

So, $\overline{x_1} + \overline{a} + \overline{-a} = \overline{x_2} + \overline{a} + \overline{-a}$

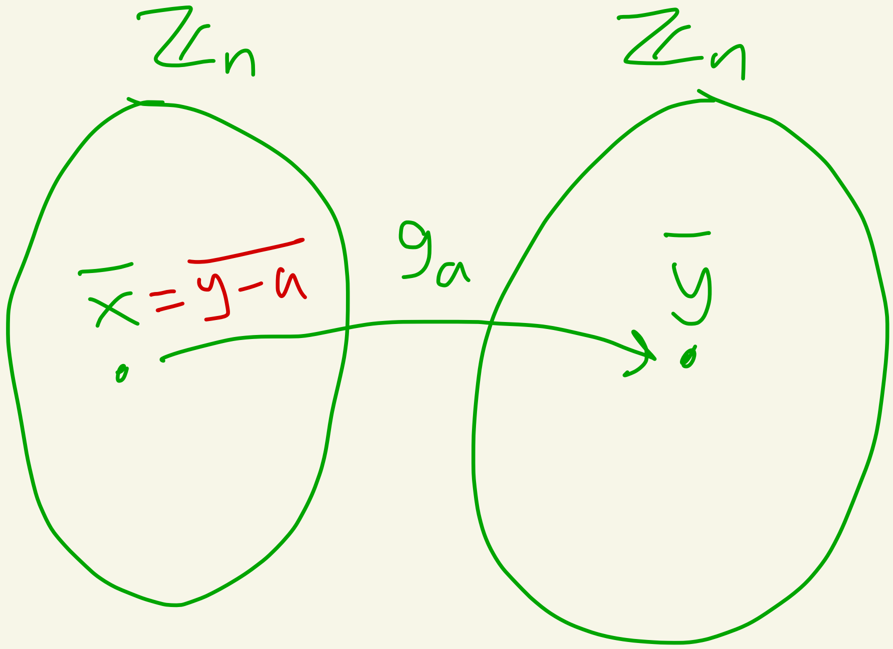
Then, $\overline{x_1} = \overline{x_2}$.

So, g_a is 1-1.

(onto)

Let $\bar{y} \in \mathbb{Z}_n$

Need to find $\bar{x} \in \mathbb{Z}_n$ where $g_a(\bar{x}) = \bar{y}$.



Need to solve $\bar{x} + \bar{a} = \bar{y}$.

Need $\bar{x} = \underbrace{\bar{y} - \bar{a}}_{\text{means } \bar{y} + \overline{-a}} = \overline{y-a}$

Verify: $g_a(\overline{y-a}) = \overline{y-a} + \bar{a} = \overline{y-a+a} = \bar{y}$

So, g_a is onto. $\blacksquare$

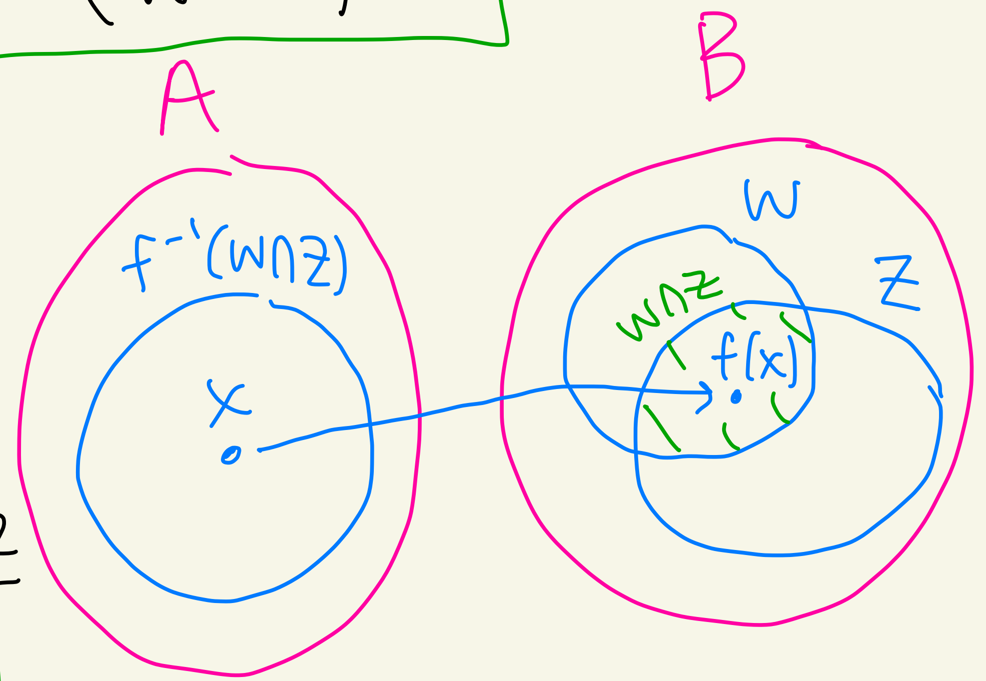
⑤ (c) $f: A \to B$, $W \subseteq B$, $Z \subseteq B$

Show: $f^{-1}(W \cap Z) = f^{-1}(W) \cap f^{-1}(Z)$

$a \in f^{-1}(C)$
$f(a) \in C$

Proof:

($\subseteq$): Let $x \in f^{-1}(W \cap Z)$



Then,

$f(x) \in W \cap Z$

So,

$f(x) \in W$ and $f(x) \in Z$

Thus,

$x \in f^{-1}(W)$ and $x \in f^{-1}(Z)$.

So,

$x \in f^{-1}(W) \cap f^{-1}(Z)$.

Thus, $f^{-1}(W \cap Z) \subseteq f^{-1}(W) \cap f^{-1}(Z)$.

$(\supseteq)$: Let $a \in f^{-1}(W) \cap f^{-1}(Z)$.

So, $a \in f^{-1}(W)$ and $a \in f^{-1}(Z)$.

Then, $f(a) \in W$ and $f(a) \in Z$.

Thus, $f(a) \in W \cap Z$.

Hence, $a \in f^{-1}(W \cap Z)$.

So, $f^{-1}(W) \cap f^{-1}(Z) \subseteq f^{-1}(W \cap Z)$.

$\blacksquare$

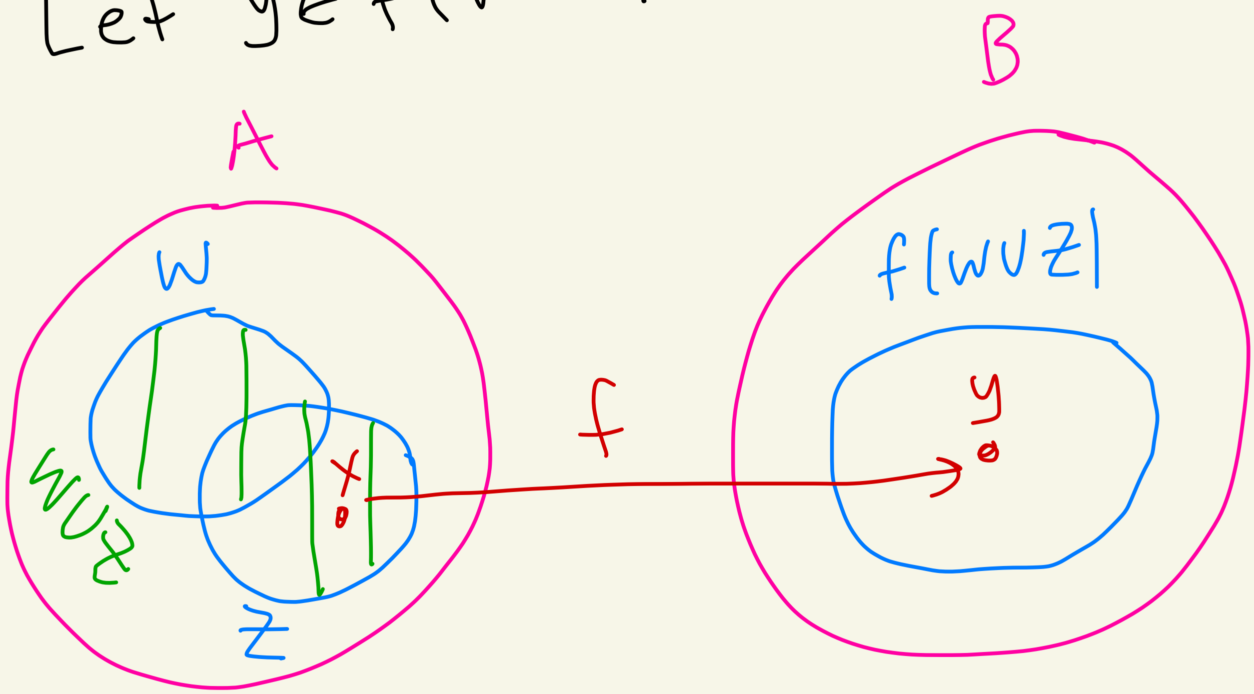
Practice Test

5A) $f: A \to B$, $W \subseteq A$, $Z \subseteq A$

Show: $f(W \cup Z) = f(W) \cup f(Z)$.

proof:

($\subseteq$): Let $y \in f(W \cup Z)$.



Then there exists $x \in W \cup Z$ where $f(x) = y$.

Since $x \in W \cup Z$ either $x \in W$ or $x \in Z$.

case 1: Suppose $x \in W$.

So, $x \in W$ and $f(x) = y$.

Then, $y \in f(W)$.

So, $y \in f(w) \cup f(Z)$.

Case 2: Suppose $x \in Z$

So, $x \in Z$ and $f(x) = y$

Then $y \in f(Z)$

So, $y \in f(w) \cup f(Z)$.

In either case, $y \in f(w) \cup f(Z)$

So, $f(W \cup Z) \subseteq f(W) \cup f(Z)$.

$(\supseteq)$: Let $y \in f(w) \cup f(Z)$.

Then, $y \in f(w)$ or $y \in f(Z)$.

Case 1: Suppose $y \in f(W)$.

Then $\exists x \in W$ where $f(x) = y$

Then, $x \in W \cup Z$ where $f(x) = y$.

So, $y \in f(W \cup Z)$

<u>case 2</u>: Suppose $y \in f(Z)$.

Then $\exists x \in Z$ where $f(x) = y$

Then $x \in W \cup Z$ where $f(x) = y$

So, $y \in f(W \cup Z)$.

In either case, $y \in f(W \cup Z)$.

So, $f(W) \cup f(Z) \subseteq f(W \cup Z)$.

$\blacksquare$