The Strong Chromatic Index of Cubic Halin Graphs

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April 19, 2011

Abstract

A *strong edge coloring* of a graph G is an assignment of colors to the edges of G such that two distinct edges are colored differently if they are incident to a common edge or share an endpoint. The *strong chromatic index* of a graph G, denoted $s\chi'(G)$, is the minimum number of colors needed for a strong edge coloring of G. A Halin graph G is a plane graph constructed from a tree T without vertices of degree two by connecting all leaves through a cycle C. If a cubic Halin graph G is different from two particular graphs Ne_2 and Ne_4, then we prove $s\chi'(G) \leqslant 7$. This solves a conjecture proposed in W. C. Shiu and W. K. Tam, The strong chromatic index of complete cubic Halin graphs, Appl. Math. Lett. 22 (2009) 754–758.

Keywords: Strong edge coloring; Strong chromatic index; Halin graph.

1 Introduction

For a graph G with vertex set $V(G)$ and edge set $E(G)$, the *line graph* $L(G)$ of G is the graph on the vertex set $E(G)$ such that two vertices in $L(G)$ are defined to be adjacent if and only if their corresponding edges in G share a common endpoint. The *distance* between two edges in G is defined to be their distance in $L(G)$. A *strong edge coloring* of a graph G is an assignment of colors to the edges of G such that two distinct edges are colored differently if they are within distance two. Thus, two edges are colored with different colors if they are incident to a common edge or share an endpoint. An *induced matching* in a graph G is the edge set of an induced subgraph of G that is also a matching. A strong edge coloring can be equivalently defined as a partition of edges into induced

*Supported in part by the National Science Council under grant NSC99-2115-M-001-004-MY3.

matchings. The *strong chromatic index* of G, denoted $s\chi'(G)$, is the minimum number of colors needed for a strong edge coloring of G.

The strong edge coloring problem is NP-complete even for bipartite graphs with girth at least 4 ([7]). However, polynomial time algorithms have been obtained for chordal graphs ([2]), co-comparability graphs ([5]), and partial k-trees ([8]).

The maximum of the degree $\deg(v)$ over all $v \in V(G)$ is written as $\Delta(G)$, or Δ when no ambiguities arise. The following outstanding conjecture was proposed by Faudree et al. [4], refining an upper bound given by Erdős and Nešetřil [3].

Conjecture 1 *For any graph G with maximum degree Δ,*

$$s\chi'(G) \leqslant \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even,} \\ \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4} & \text{if } \Delta \text{ is odd.} \end{cases}$$

It is straightforward to see that Conjecture 1 holds when $\Delta \leqslant 2$. Conjecture 1 was proved to be true for $\Delta = 3$ by Andersen [1] and, independently, by Horák et al. [6]. It remains open when $\Delta \geqslant 4$.

A *Halin graph* is a plane graph G constructed as follows. Let T be a tree having at least 4 vertices, called the *characteristic tree* of G. All vertices of T are either of degree 1, called *leaves*, or of degree at least 3. Let C be a cycle, called the *adjoint cycle* of G, connecting all leaves of T in such a way that C forms the boundary of the unbounded face. We usually write $G = T \cup C$ to reveal the characteristic tree and the adjoint cycle.

For $n \geqslant 3$, the *wheel* W_n is a particular Halin graph whose characteristic tree is the complete bipartite graph $K_{1,n}$. A graph is said to be *cubic* if the degree of every vertex is 3. For $h \geqslant 1$, a cubic Halin graph Ne_h, called a *necklace*, was constructed in [9]. Its characteristic tree T_h consists of the path $v_0, v_1, \ldots, v_h, v_{h+1}$ and leaves $v_1', v_2', \ldots, v_h'$ such that the unique neighbor of v_i' in T_h is v_i for $1 \leqslant i \leqslant h$ and vertices $v_0, v_1', \ldots, v_h', v_{h+1}$ are in order to form the adjoint cycle C_{h+2}. The strong chromatic index of a cubic Halin graph is easily seen to be at least 6. The following upper bound was conjectured in Shiu and Tam [10].

Conjecture 2 *If G is a cubic Halin graph that is different from any necklace, then $s\chi'(G) \leqslant 7$.*

We shall prove the validity of this conjecture.

2 Main result

Since the line graph of a cycle C_n of n vertices is C_n itself and any edge of the characteristic tree of a wheel is within distance 2 to any edge of the adjoint cycle, it is straightforward to obtain the following two lemmas.

Lemma 3 *For the cycle C_n, we have*

$$s\chi'(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3, \\ 5 & \text{if } n = 5, \\ 4 & \text{otherwise.} \end{cases}$$

Lemma 4 *For the wheel W_n, we have*

$$s\chi'(W_n) = \begin{cases} n+3 & \text{if } n \equiv 0 \pmod 3, \\ n+5 & \text{if } n = 5, \\ n+4 & \text{otherwise.} \end{cases}$$

The strong chromatic index of a necklace was determined in [9] as follows.

Lemma 5 *Suppose $h \geqslant 1$.*

$$s\chi'(Ne_h) = \begin{cases} 6 & \text{if } h \text{ is odd,} \\ 7 & \text{if } h \geqslant 6 \text{ and is even,} \\ 8 & \text{if } h = 4, \\ 9 & \text{if } h = 2. \end{cases}$$

Theorem 6 *If a cubic Halin graph $G = T \cup C$ is different from Ne_2 and Ne_4, then $s\chi'(G) \leqslant 7$.*

Proof. We prove the theorem by induction on the length m of the adjoint cycle C. It is easy to see that the only cubic Halin graphs with $m = 3, 4$, and 5 are W_3, Ne_2, and Ne_3, respectively. They all satisfy our theorem by Lemmas 4 and 5. Now assume $m \geqslant 6$.

In our later inductive steps, we use two basic operations to reduce a cubic Halin graph G to another cubic Halin graph G' such that the length of the adjoint cycle of G' is shorter than that of G. If G' is equal to neither Ne_2 nor Ne_4, then $s\chi'(G') \leqslant 7$ by the induction hypothesis. Otherwise, up to symmetry, G belongs to a list of eleven cubic Halin graphs, each of which can have a strong edge coloring using at most seven colors. All such colorings are supplied in Figure 4 in the Appendix.

Let $P : u_0, u_1, \ldots, u_l$, $l \geqslant 5$, be a longest path in T. Since P is of maximum length, all neighbors of u_1, except u_2, are leaves. We may change notation to let $w = u_3$, $u = u_2$, $v = u_1$, and v_1 and v_2, be the neighbors of v on C as depicted in Figure 1.

Since $\deg(u) = 3$, there exists a path Q from u to x_1 or y_1 with $P \cap Q = \{u\}$. Without loss of generality, we may assume that Q is a path from u to y_1. Since P is a longest path in T, Q has length at most two. It follows that $uy_3 \in E(T)$ or $u = y_3$. The former implies $y_2y_3 \in E(T)$ and the latter means $uy_1 \in E(T)$.

Case 1. $uy_3 \in E(T)$.

Consider Figure 2. Now let G' be the graph obtained from G by deleting v, v_1, v_2, y_1, y_2, y_3, and adding two new edges ux_1 and uz. By the induction hypothesis, we may assume that there exists a strong edge coloring f for $E(G')$ using colors from the set

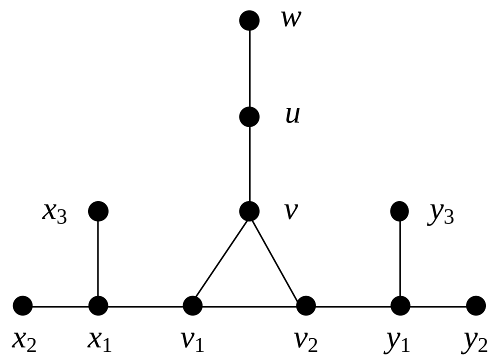

Figure 1: Around the end of a longest path in the characteristic tree.

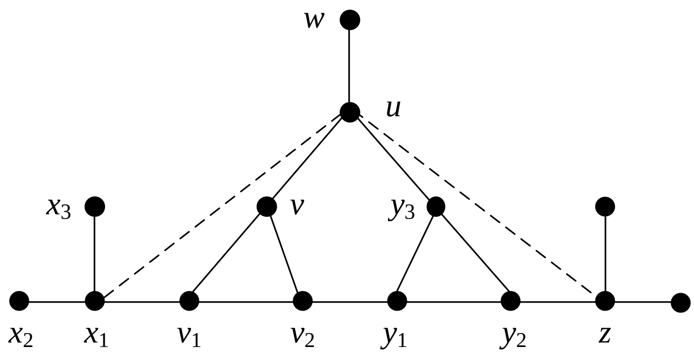

Figure 2: The case $uy_3 \in E(T)$.

$[7] = \{1, 2, \ldots, 7\}$. Without loss of generality, we assume that $f(wu) = 1$, $f(ux_1) = 2$, $f(uz) = 3$. Except the edge ux_1, let the other two edges in G' incident to x_1 be colored with t_1 and t_2. Except the edge uz, let the other two edges in G' incident to z be colored with s_1 and s_2. Note that $\{s_1, s_2, t_1, t_2\} \cap \{1, 2, 3\} = \emptyset$. Now we shall extend f to the remaining edges of G to get a strong edge coloring using seven colors. We first let $f(v_2y_1) = 1$, $f(uy_3) = f(x_1v_1) = 2$ and $f(uv) = f(y_2z) = 3$.

Subcase 1 $\{s_1, s_2\} = \{t_1, t_2\}$.

Let $\{\alpha, \beta\} = [7] \setminus \{1, 2, 3, t_1, t_2\}$. Let $f(vv_2) = t_1$, $f(y_1y_3) = t_2$, $f(vv_1) = f(y_1y_2) = \alpha$, $f(v_1v_2) = f(y_2y_3) = \beta$.

Subcase 2 $\{s_1, s_2\} \cap \{t_1, t_2\} = \emptyset$.

Let $f(vv_2) = f(y_2y_3) = t_1$, $f(y_1y_2) = t_2$, $f(vv_1) = f(y_1y_3) = s_1$, $f(v_1v_2) = s_2$.

Subcase 3 $s_1 = t_1$ and $s_2 \neq t_2$.

Let $\{\alpha\} = [7] \setminus \{1, 2, 3, s_1, s_2, t_2\}$. Let $f(vv_2) = s_1$, $f(vv_1) = f(y_1y_3) = s_2$, $f(y_1y_2) = t_2$, $f(v_1v_2) = f(y_2y_3) = \alpha$.

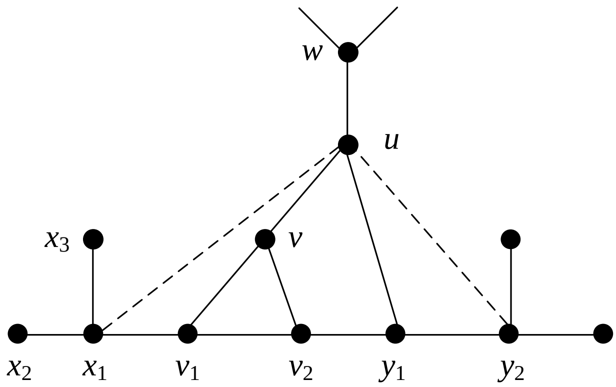

Figure 3: The case $u = y_3$.

Case 2. $u = y_3$.

Consider Figure 3. Let G' be the graph obtained from G by deleting v, v_1, v_2, y_1, and adding two new edges ux_1 and uy_2. By the induction hypothesis, we may assume that there exists a strong edge coloring f for $E(G')$ using colors from the set [7]. Without loss of generality, assume that $f(ux_1) = 1$, $f(uy_2) = 2$, $f(uw) = 3$, $f(x_1x_2) = 4$, and $f(x_1x_3) = 5$. Except the edge uw, let the other two edges in G' incident to w be colored with t_1 and t_2. Except the edge vy_2, let the other two edges in G' incident to y_2 be colored with s_1 and s_2. Note that $\{s_1, s_2, t_1, t_2\} \cap \{1, 2, 3\} = \emptyset$. Now we shall extend f to the remaining edges of G to get a strong edge coloring using seven colors. We first let $f(x_1v_1) = f(uy_1) = 1$, $f(vv_1) = f(y_1y_2) = 2$, and $f(v_1v_2) = 3$. There are five colors $1, 2, 3, t_1, t_2$ forbidden for the edge uv, hence $f(uv)$ can be defined. Next, there are at most six colors $1, 2, 3, s_1, s_2, f(uv)$ forbidden for the edge v_2y_1, hence $f(v_2y_1)$ can be defined. Finally, there are five colors $1, 2, 3, f(uv), f(v_2y_1)$ forbidden for the edge vv_2, hence $f(vv_2)$ can be defined. ∎

Appendix

Figure 4 is a list of eleven basic graphs each of which is depicted with a strong edge coloring using seven colors. The white vertices of a graph are to be deleted during the inductive step so that the reduced graph becomes Ne_2 or Ne_4.

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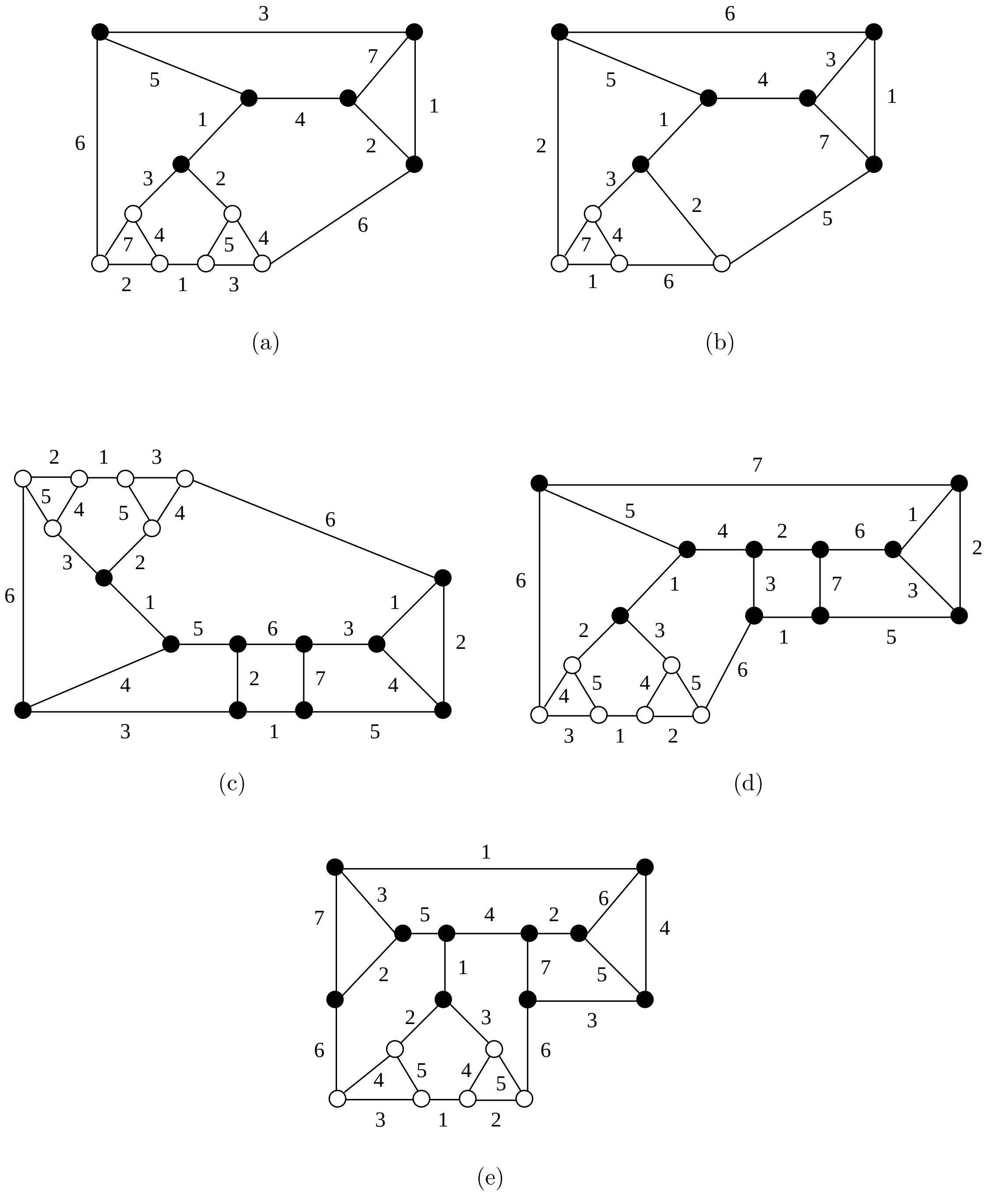

Figure 4: Eleven basic cubic Halin graphs.

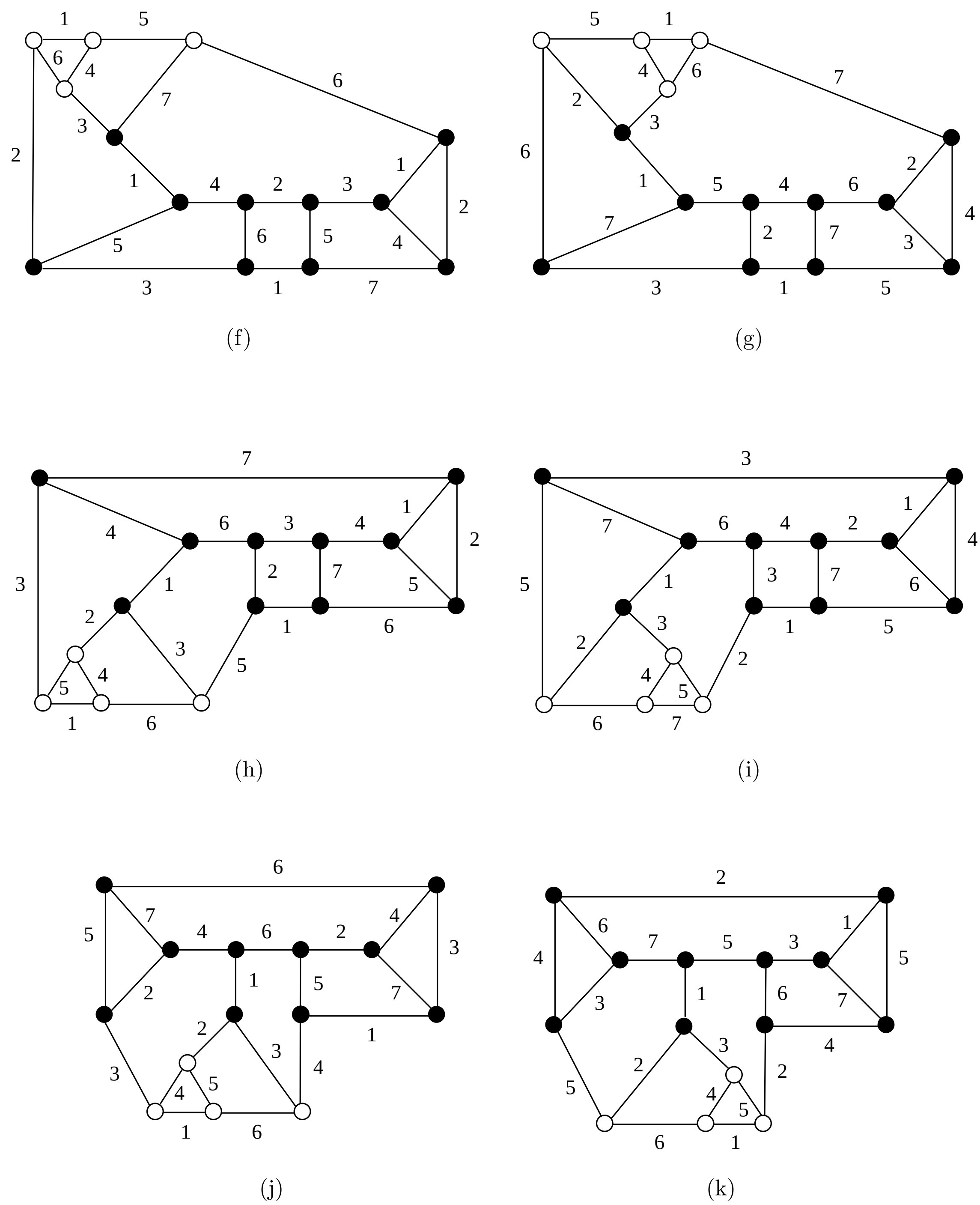

Figure 4: Eleven basic cubic Halin graphs.