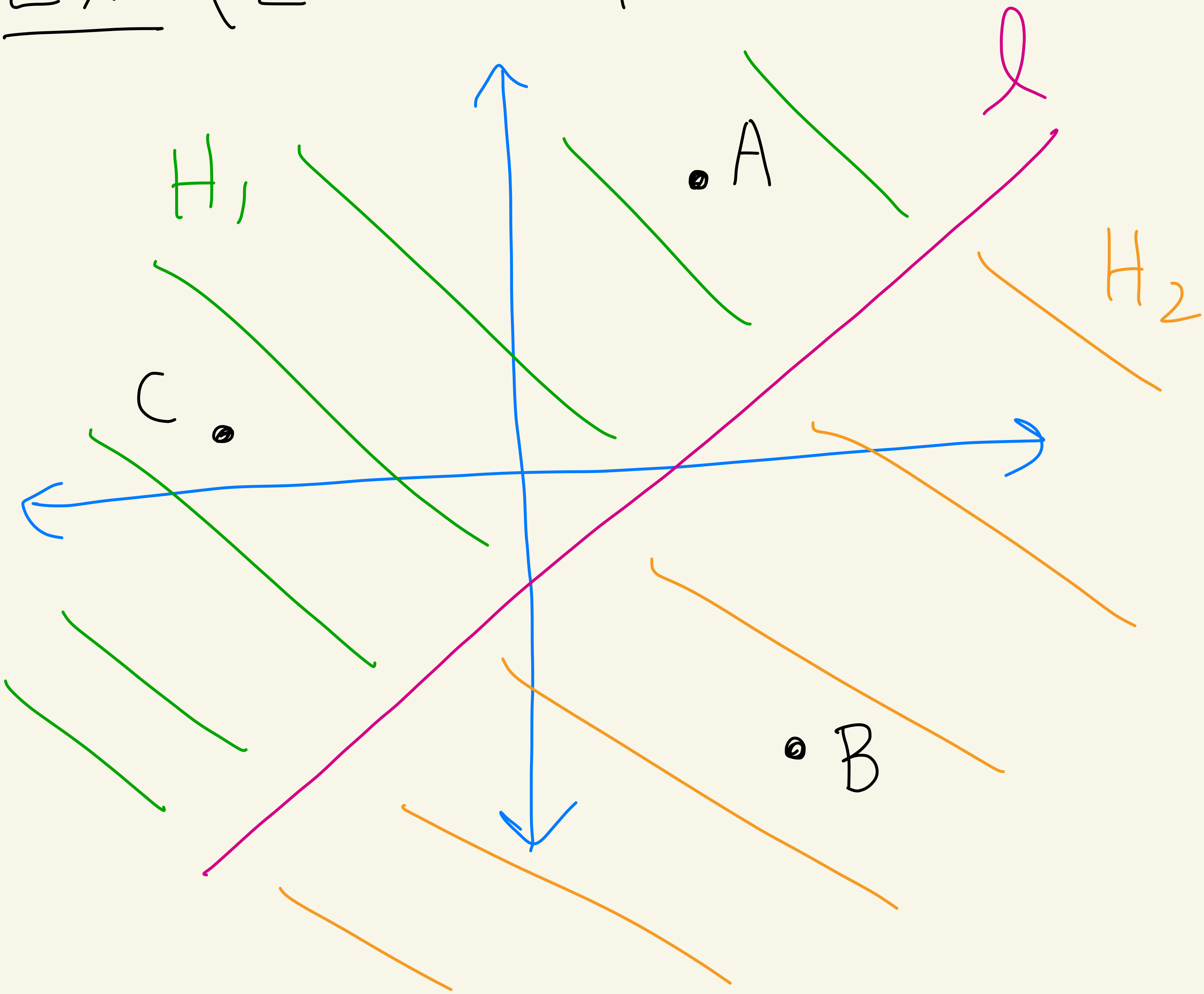
Math 4300
10/18/23

Ex: (Euclidean plane)



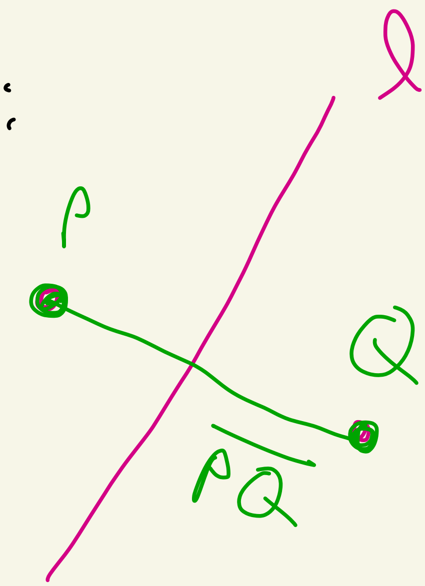
$A, C \in H_1$, so A, C are on the same side of ℓ.

$A \in H_1$ and $B \in H_2$ so A, B are on opposite sides of ℓ.

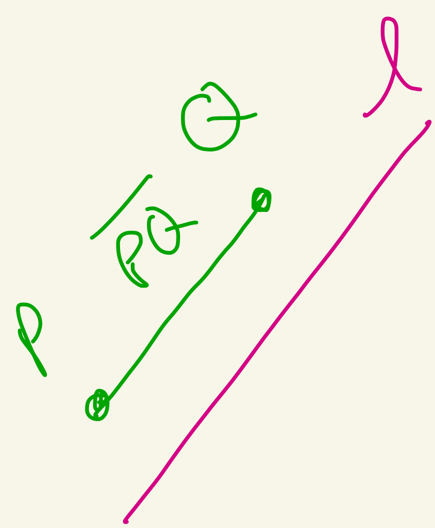
Three theorems from HW 7

Theorem: Let $(\mathcal{P}, \mathcal{L}, d)$ be a metric geometry that satisfies PSA. Let $\ell \in \mathcal{L}$ be a line. Let $P, Q \in \mathcal{P}$ with $P \notin \ell$ and $Q \notin \ell$. Then:

(i) P and Q are on opposite sides of ℓ iff $\overline{PQ} \cap \ell \neq \emptyset$



(ii) P and Q are on the same side of ℓ iff $\overline{PQ} \cap \ell = \emptyset$



Proof: HW 7 #6 $\blacksquare$

Theorem: Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry that satisfies PSA. Let $P, Q, R \in \mathscr{P}$ and $\ell \in \mathscr{L}$.

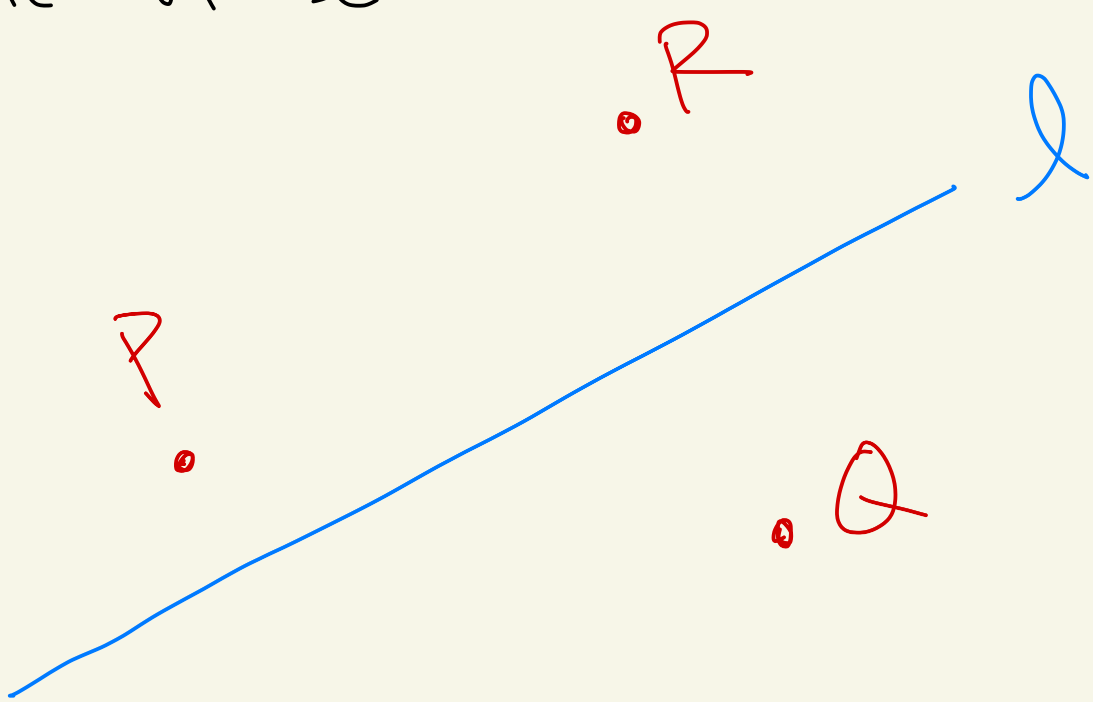
If P and Q are on opposite sides of ℓ and Q and R are on opposite sides of ℓ, then P and R are on the same side of ℓ.

proof:

HW 7

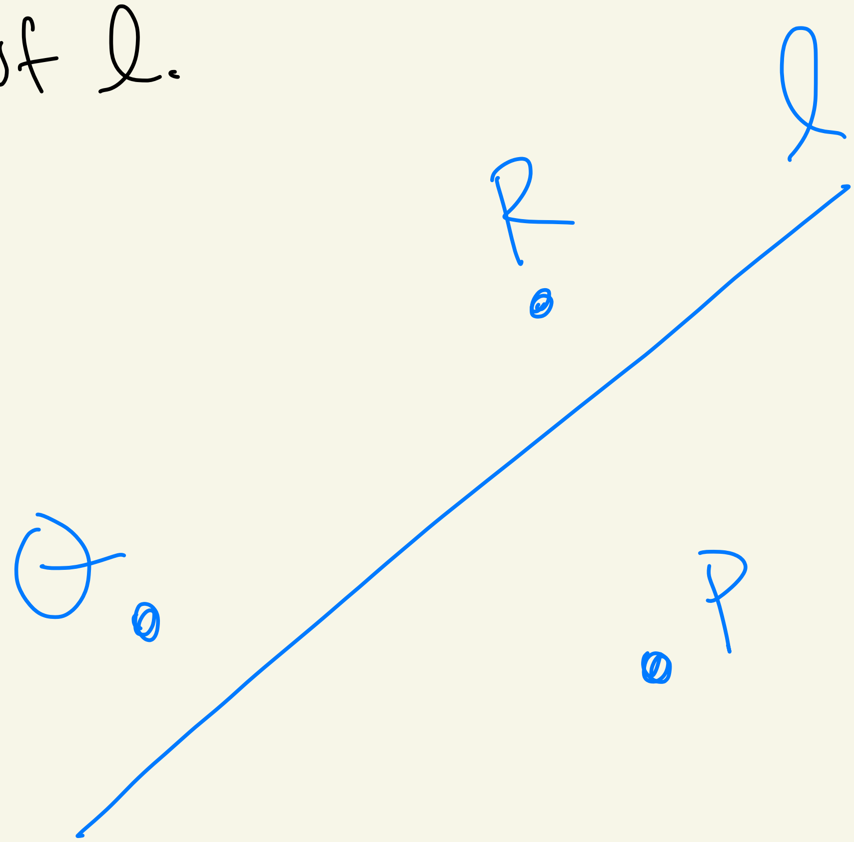
#7

$\blacksquare$



Theorem: Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry that satisfies PSA. Let $P, Q, R \in \mathscr{P}$ and $\ell \in \mathscr{L}$.

If P and Q are on opposite sides of ℓ and Q and R are on the same side of ℓ, then P and R are on opposite sides of ℓ.



proof:

HW 7

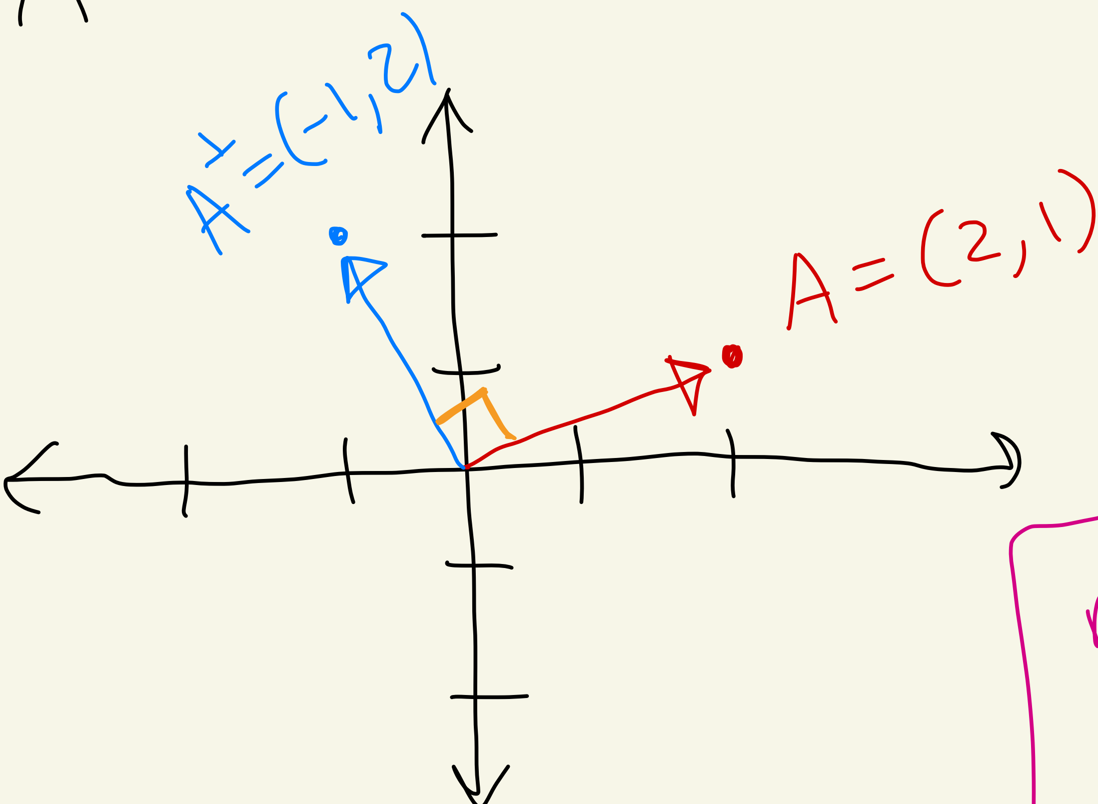
#8

$\blacksquare$

Our next goal is to show that the Euclidean plane satisfies the PSA axioms.

<u>Def:</u> If $A = (x,y) \in \mathbb{R}^2$, then define $A^{\perp} = (-y, x)$

$A^{\perp}$ is read "A perp".



Idea:

We get $A^{\perp}$ by rotating A by $90°$

Theorem:

(i) If $X \in \mathbb{R}^2$, then $\langle X, X^\perp \rangle = 0$

(ii) Let $X, Z \in \mathbb{R}^2$ and $X \neq (0,0)$.
If $\langle Z, X^\perp \rangle = 0$, then
$Z = tX$ for some $t \in \mathbb{R}$.

proof:

(i) Let $X = (a,b)$.

Then,

$$\langle X, X^\perp \rangle = \langle (a,b), (-b,a) \rangle$$

$$= (a)(-b) + (b)(a) = 0$$

(ii) Let $X = (a,b) \neq (0,0)$
and $Z = (c,d)$.

Suppose $\langle Z, X^{\perp} \rangle = 0$.

Then, $0 = \langle Z, X^{\perp} \rangle = \langle (c,d), (-b,a) \rangle$

$= -cb + da$

So, $\boxed{ad - bc = 0.}$ (*)

Since $X = (a,b) \neq (0,0)$, either $a \neq 0$ or $b \neq 0$.

Case 1: Suppose $a \neq 0$.

Then from (*) we get $d = \frac{bc}{a}$.

Thus,

$$Z = (c,d) = \left(c, \frac{bc}{a}\right) = \frac{c}{a} \cdot (a,b)$$
$$= \frac{c}{a} \cdot X = tX$$

where $t = \frac{c}{a}$.

<u>case 2</u>: Suppose $b \neq 0$.

Then from $(*)$ we get $c = \frac{ad}{b}$.

Thus,

$$Z = (c, d) = \left(\frac{ad}{b}, d\right) = \frac{d}{b} \cdot (a, b)$$

$$= \frac{d}{b} \cdot X = tX$$

where $t = \frac{d}{b}$.

In either case, $Z = tX$ for some $t \in \mathbb{R}$. $\blacksquare$

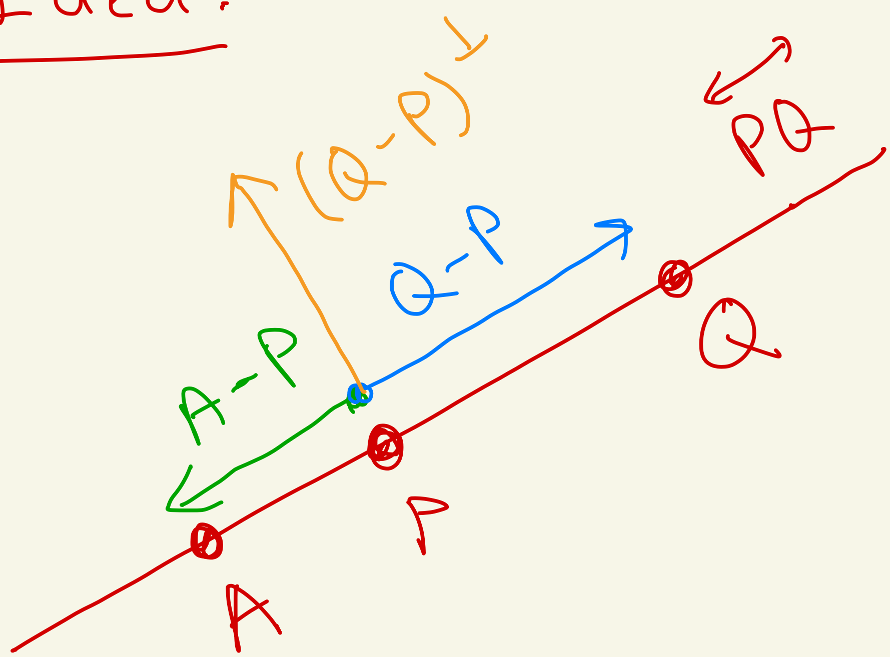
Theorem: Consider the Euclidean plane $\mathcal{E} = (\mathbb{R}^2, \mathcal{L}_E, d_E)$.

Let $P, Q \in \mathbb{R}^2$ where $P \neq Q$.

Then

$$\overleftrightarrow{PQ} = \{ A \in \mathbb{R}^2 \mid \langle A - P, (Q-P)^{\perp} \rangle = 0 \}$$

Idea:



Want $A-P$ to be perpendicular to $(Q-P)^{\perp}$

proof:

$\boxed{\subseteq}$: Let $A \in \overleftrightarrow{PQ}$.

Then from Topic 3 we have

$$A = P + t(Q-P)$$

where $t \in \mathbb{R}$.

Then,

$$\langle (A-P), (Q-P)^{\perp} \rangle$$

$$= \langle t(Q-P), (Q-P)^{\perp} \rangle$$

$$= t \langle (Q-P), (Q-P)^{\perp} \rangle$$

$$= t \cdot 0 = 0$$

$\langle t\,a, b \rangle = t \langle a, b \rangle$

previous thm

Thus,

$$\overleftrightarrow{PQ} \subseteq \{A \in \mathbb{R}^2 \mid \langle (A-P), (Q-P)^{\perp} \rangle = 0\}.$$

$\boxed{\supseteq}$: Now let $A \in \mathbb{R}^2$ with $\langle (A-P), (Q-P)^{\perp} \rangle = 0$.

Since $P \neq Q$, we know $Q-P \neq (0,0)$.
So, by the previous theorem

$$A - P = t(Q-P)$$

where $t \in \mathbb{R}$.
Then $A = P + t(Q-P)$.
By Topic 3, we get $A \in \overleftrightarrow{PQ}$.

So,

$$\{A \in \mathbb{R}^2 \mid \langle (A-P), (Q-P)^{\perp} \rangle = 0\} \subseteq \overleftrightarrow{PQ}.$$

$\blacksquare$