Math 4300

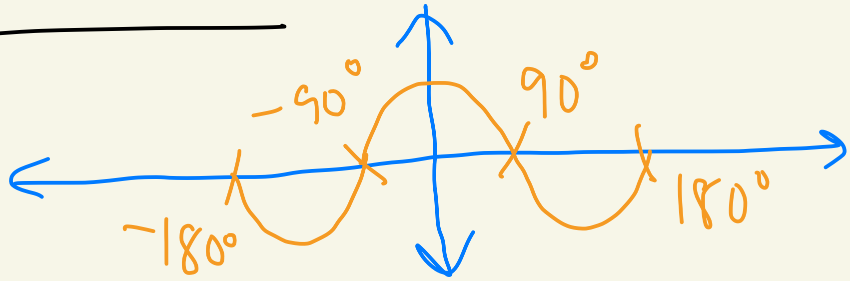
10/23/23

Test 2 –

Remove HW 6 from study guide.

HW 3, 4, 5 on test 2

(Topic 7 continued...)



Idea:

Recall from Calculus that

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|} = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \cdot \|\vec{w}\|}$$

where θ is the angle between $\vec{v}$ and $\vec{w}$.

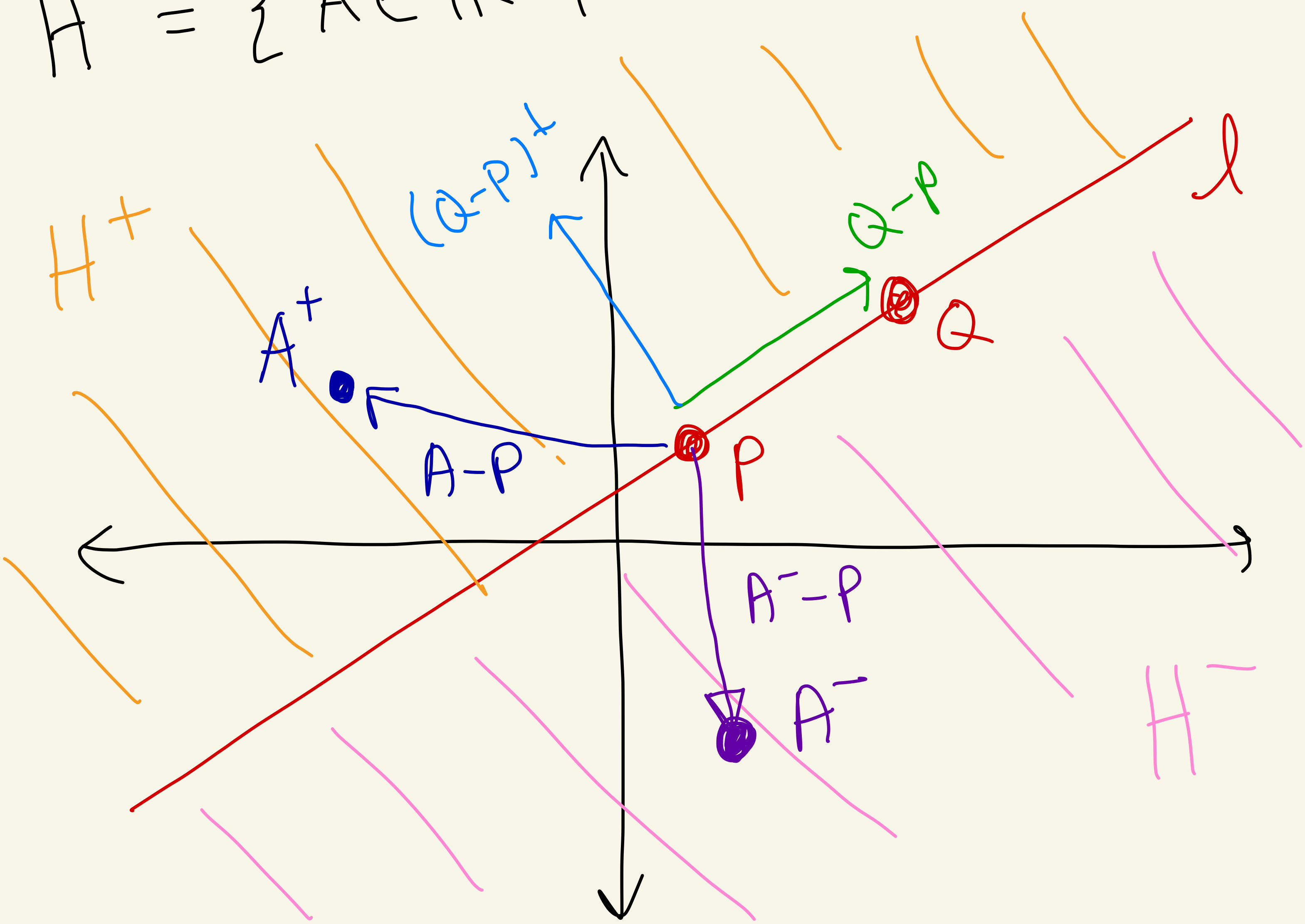
So, $\langle \vec{v}, \vec{w} \rangle > 0$ iff $\underbrace{-90^\circ < \theta < 90^\circ}_{\cos(\theta) > 0}$

And $\langle \vec{v}, \vec{w} \rangle < 0$ iff $\theta < -90^\circ$ or $90^\circ < \theta$.

Def: Consider the Euclidean plane $\mathcal{E} = (\mathbb{R}^2, \mathcal{L}_E, d_E)$.

Let $l = \overleftrightarrow{PQ}$. Define

$$H^+ = \{ A \in \mathbb{R}^2 \mid \langle A-P, (Q-P)^\perp \rangle > 0 \}$$
$$H^- = \{ A \in \mathbb{R}^2 \mid \langle A-P, (Q-P)^\perp \rangle < 0 \}$$



In pic: $A^+ \in H^+$ and $A^- \in H^-$

Theorem: Let $\ell = \overleftrightarrow{PQ}$ in the Euclidean plane. Let H^+ and H^- be defined as above. Then H^+ and H^- are convex.

Proof: Let's prove this for H^+. The H^- case is similar.

Let $A, B \in H^+$.

Since $A, B \in H^+$ we know

$$\langle A-P, (Q-P)^\perp \rangle > 0$$

and $\langle B-P, (Q-P)^\perp \rangle > 0$

We need to show that $\overline{AB} \subseteq H^+$

Let $C \in \overline{AB}$. $\leftarrow$ [So either $C=A$, $C=B$, or $A-C-B$]

We need to show $C \in H^+$.

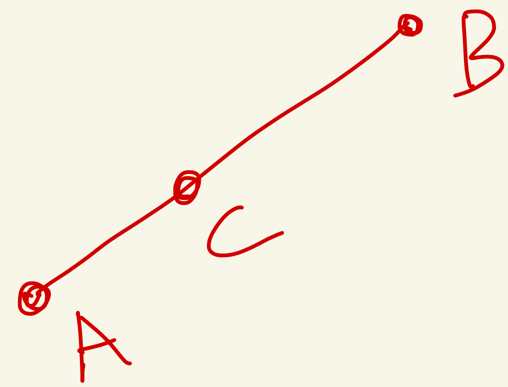
If $C = A$ or $C = B$, then $C \in H^+$.

So suppose $A - C - B$.

From HW 4 #10 we know

$$C = A + t(B - A)$$

where $0 < t < 1$.



Thus,

$$\langle (C-P), (Q-P)^\perp \rangle$$

$$= \langle \underbrace{A + t(B-A)}_{C} - P, (Q-P)^\perp \rangle$$

$$= \langle (1-t)A + Bt - P, (Q-P)^\perp \rangle$$

$$= \langle (1-t)(A-P) + t(B-P), (Q-P)^{\perp} \rangle$$

$$= \langle (1-t)(A-P), (Q-P)^{\perp} \rangle + \langle t(B-P), (Q-P)^{\perp} \rangle$$

$$\boxed{\langle M+N, L \rangle = \langle M, L \rangle + \langle N, L \rangle}$$

$$\boxed{\langle cM, N \rangle = c\langle M, N \rangle}$$

$$= \underbrace{(1-t)}_{\substack{0<t<1 \\ 0<1-t}} \underbrace{\langle A-P, (Q-P)^{\perp} \rangle}_{>0 \text{ since } A \in H^{+}} + \underbrace{t}_{t>0} \underbrace{\langle B-P, (Q-P)^{\perp} \rangle}_{>0 \text{ since } B \in H^{+}} > 0$$

Thus $\langle C-P, (Q-P)^{\perp} \rangle > 0$

and $C \in H^{+}$.

Thus, $\overline{AB} \subseteq H^{+}$. $\blacksquare$

<u>Theorem:</u> The Euclidean plane $\mathcal{E} = (\mathbb{R}^2, \mathcal{L}_E, d_E)$ satisfies the PSA.

<u>proof:</u> Let $P, Q \in \mathbb{R}^2$ where $P \neq Q$. Let $\ell = \overleftrightarrow{PQ}$.

Let

$$H^+ = \{A \in \mathbb{R}^2 \mid \langle A-P, (Q-P)^\perp \rangle > 0\}$$
$$H^- = \{A \in \mathbb{R}^2 \mid \langle A-P, (Q-P)^\perp \rangle < 0\}$$

Recall

$$\ell = \overleftrightarrow{PQ} = \{A \in \mathbb{R}^2 \mid \langle A-P, (Q-P)^\perp \rangle = 0\}$$

Thus, $\mathbb{R}^2$ breaks up into 3 disjoint sets:

$$\mathbb{R}^2 = H^+ \cup H^- \cup \ell$$

So,

(i) $\mathbb{R}^2 - \ell = H^+ \cup H^-$

(ii) $H^+ \cap H^- = \phi$.

The last theorem showed

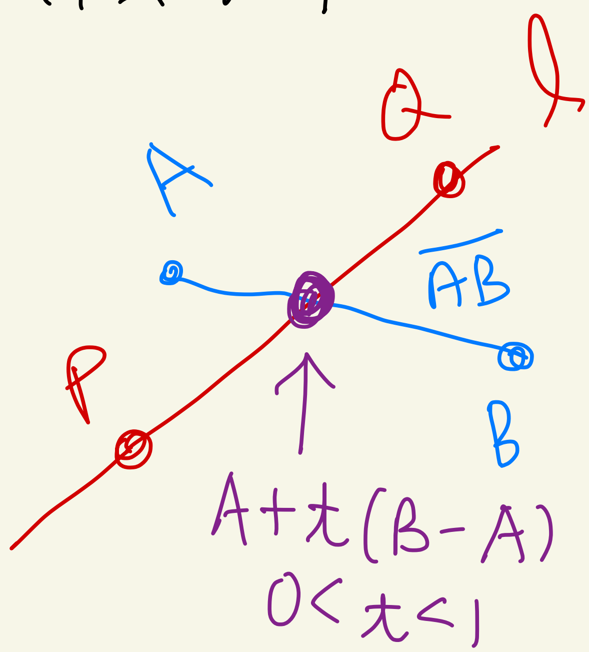
(iii) H^+ and H^- are convex.

So we just have to show (iv).

Let $A \in H^+$ and $B \in H^-$.

We need to show $\overline{AB} \cap \ell \neq \phi$.

To do this we need to find $t \in \mathbb{R}$ with $0 < t < 1$ where $A + t(B-A) \in \ell$



We need to solve

$$\langle (A+t(B-A)-P, (Q-P)^{\perp} \rangle = 0$$

for $0 < t < 1$.

$\langle M+N, L \rangle = \langle M, L \rangle + \langle N, L \rangle$

This equation can be rewritten as

$$\langle A-P, (Q-P)^{\perp} \rangle + \langle t(B-A), (Q-P)^{\perp} \rangle = 0$$

or

$$\langle A-P, (Q-P)^{\perp} \rangle = -\langle t(B-A), (Q-P)^{\perp} \rangle$$

or

$$\langle A-P, (Q-P)^{\perp} \rangle = \langle -t(B-A), (Q-P)^{\perp} \rangle$$

$c\langle M, N \rangle = \langle cM, N \rangle$

or

$$\boxed{\langle A-P, (Q-P)^{\perp} \rangle = t\langle A-B, (Q-P)^{\perp} \rangle \quad (*)}$$

We want to solve for t.

Is $\langle A-B, (Q-P)^{\perp} \rangle \neq 0$?

Well,

$$\langle A-B, (Q-P)^{\perp} \rangle \qquad (**)$$

$$= \langle (A-P)-(B-P), (Q-P)^{\perp} \rangle$$

$$= \underbrace{\langle A-P, (Q-P)^{\perp} \rangle}_{>0 \text{ since } A \in H^{+}} - \underbrace{\langle B-P, (Q-P)^{\perp} \rangle}_{<0 \text{ since } B \in H^{-}}$$

$$> 0.$$

Thus in $(*)$ we get

$$t = \frac{\langle A-P, (Q-P)^{\perp} \rangle}{\langle A-B, (Q-P)^{\perp} \rangle} \qquad (***)$$

So, $t > 0$ since $\langle A-P, (Q-P)^{\perp} \rangle > 0$ and $\langle A-B, (Q-P)^{\perp} \rangle > 0$.

Is $t < 1$?

Yes because (**) says that

$\langle A-B, (Q-P)^{\perp} \rangle > \langle A-P, (Q-P)^{\perp} \rangle$

So the above t works!

Thus, $\overline{AB} \cap \ell \neq \phi$.

$\blacksquare$

<u>Theorem:</u> The hyperbolic plane $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H, d_H)$ satisfies PSA.

<u>proof:</u> See Millman/Parker page 73

$\blacksquare$
