Math 4300
10/4/23

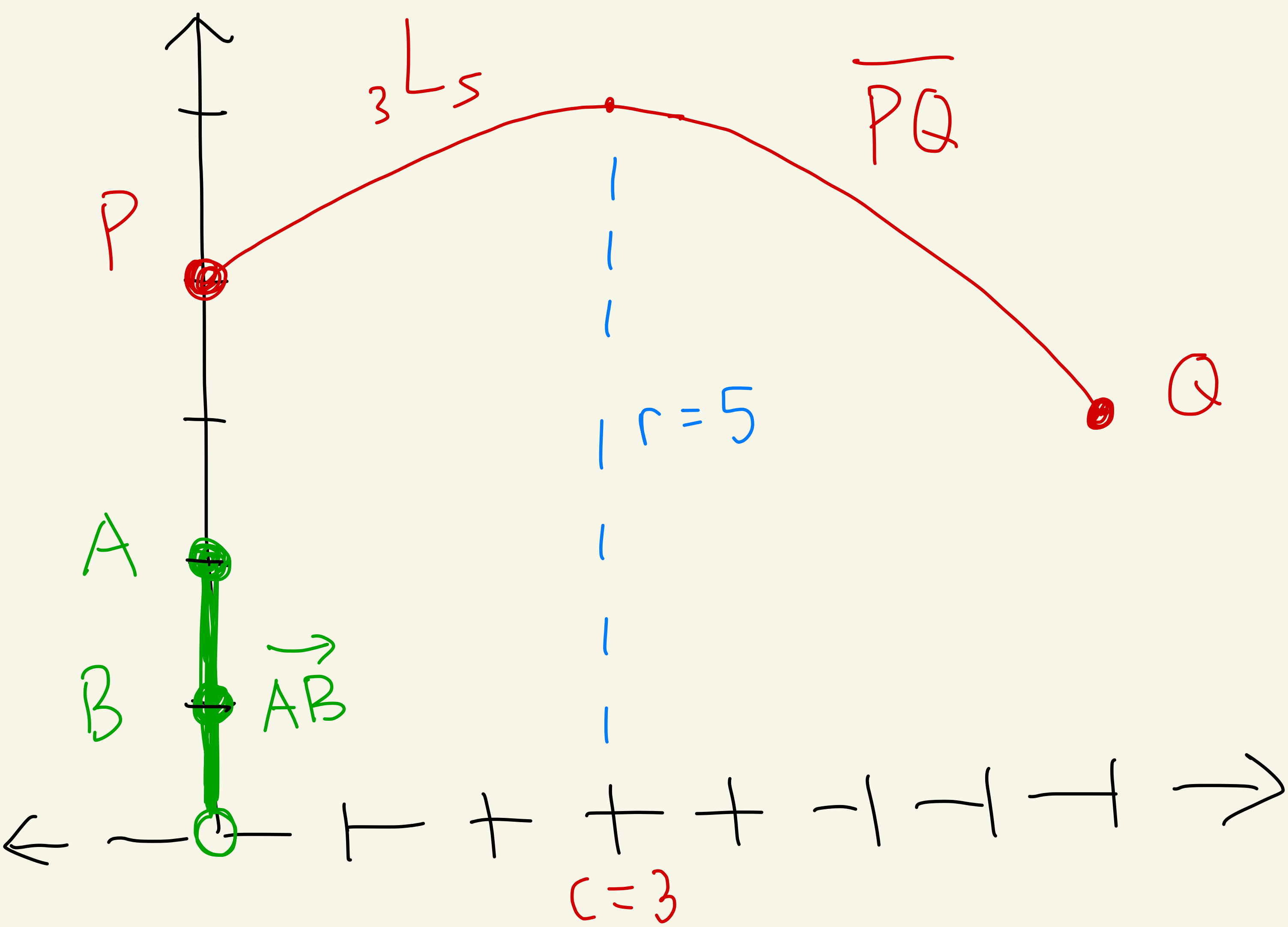
Ex: In the hyperbolic plane

$\mathcal{H} = (\mathbb{H}, \mathcal{L}_H, d_H)$, let

$A = (0,2)$, $B = (0,1)$,

$P = (0,4)$, $Q = (7,3)$

Find $C \in \overrightarrow{AB}$ so that $\overline{AC} \simeq \overline{PQ}$



Step 1: We want to calculate $\underbrace{PQ}_{d_H(P,Q)}$.

We need $\overleftrightarrow{PQ}$.

Plug $P=(0,4)$ and $Q=(7,3)$
into $(x-c)^2+y^2=r^2$.

We get

$$\boxed{\begin{aligned}(0-c)^2+(4)^2&=r^2 \leftarrow P\\ (7-c)^2+(3)^2&=r^2 \leftarrow Q\end{aligned}}$$

$$\Downarrow$$

$$\boxed{\begin{aligned}c^2 \qquad +16&=r^2 \quad (1)\\ c^2-14c+58&=r^2 \quad (2)\end{aligned}}$$

$$\Downarrow$$

$$\boxed{c^2+16=r^2=c^2-14c+58}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} 16 &= -14c + 58 \\ 14c &= 42 \\ c &= \frac{42}{14} = 3 \end{aligned}}$$

$\Downarrow$

Plug $c = 3$ into ① to get

$$r^2 = c^2 + 16 = 9 + 16 = 25$$

$$r = 5$$

So, $\overleftrightarrow{PQ} = {}_cL_r = {}_3L_5$.

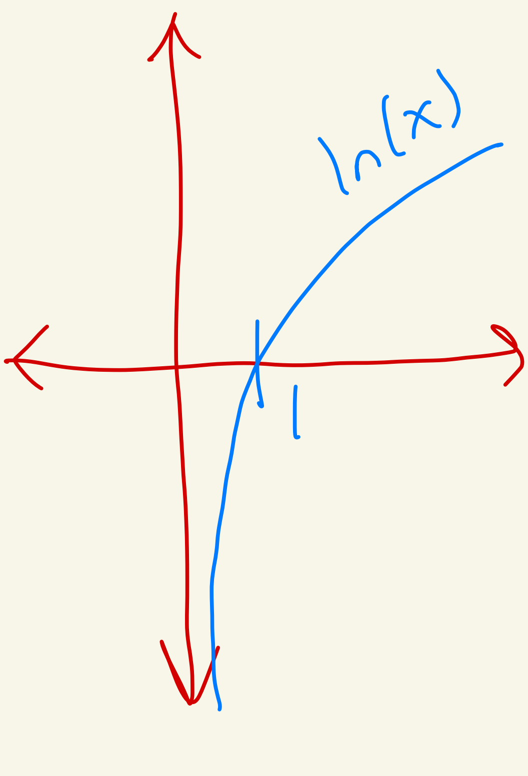
Recall on ${}_cL_r$ we have

$$d_H((x_1, y_1), (x_2, y_2)) = \left| \ln \left(\frac{\frac{x_1 - c + r}{y_1}}{\frac{x_2 - c + r}{y_2}} \right) \right|$$

Then,

$$PQ = d_H\big((0,4),(7,3)\big) = \left|\ln\left(\frac{\frac{0-3+5}{4}}{\frac{7-3+5}{3}}\right)\right|$$

$$= \left|\underbrace{\ln\left(\frac{1}{6}\right)}_{\ln\left(\frac{1}{6}\right)<0}\right| = -\ln\left(\frac{1}{6}\right)$$



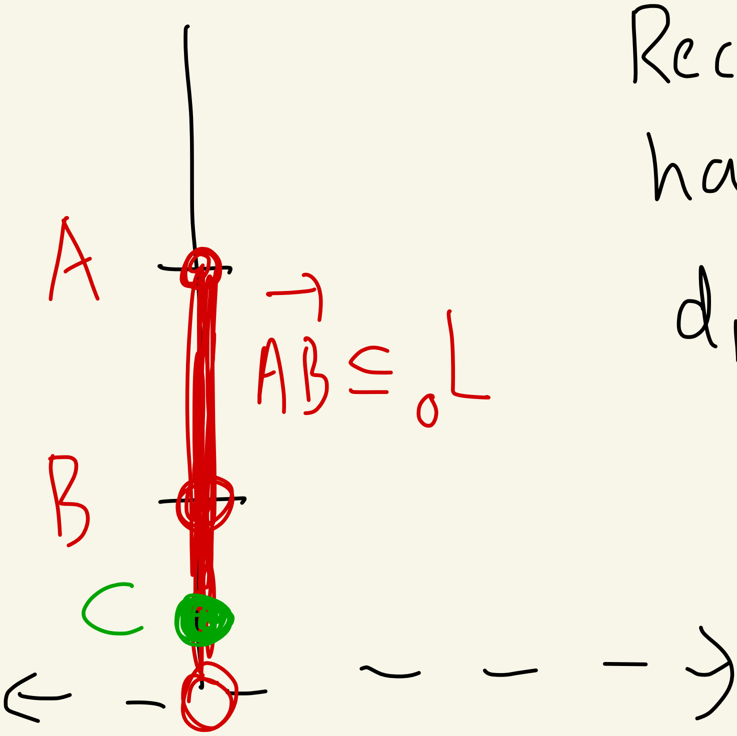
$$= \ln\left(\left(\frac{1}{6}\right)^{-1}\right)$$

$$\boxed{-\ln(A) = \ln(A^{-1})}$$

$$= \boxed{\ln(6)}$$

$$\approx 1.79$$

Now we need to find $C \in \overrightarrow{AB}$ where $AC = \ln(6)$.



Recall on ${}_aL$ we have

$$d_H((x_1,y_1),(x_2,y_2)) = \left|\ln\left(\frac{y_1}{y_2}\right)\right|$$

We know $C = (0,y)$.

We need to solve for y in:

$$\ln(6) = d_H(A,C) = d_H((0,2),(0,y))$$

$$= \underbrace{\left|\ln\left(\frac{2}{y}\right)\right|}_{y<2} = \ln\left(\frac{2}{y}\right)$$

$$1 < \frac{2}{y}$$

$$\ln\left(\frac{2}{y}\right) > 0$$

Need $\ln(6) = \ln\left(\frac{2}{y}\right)$.

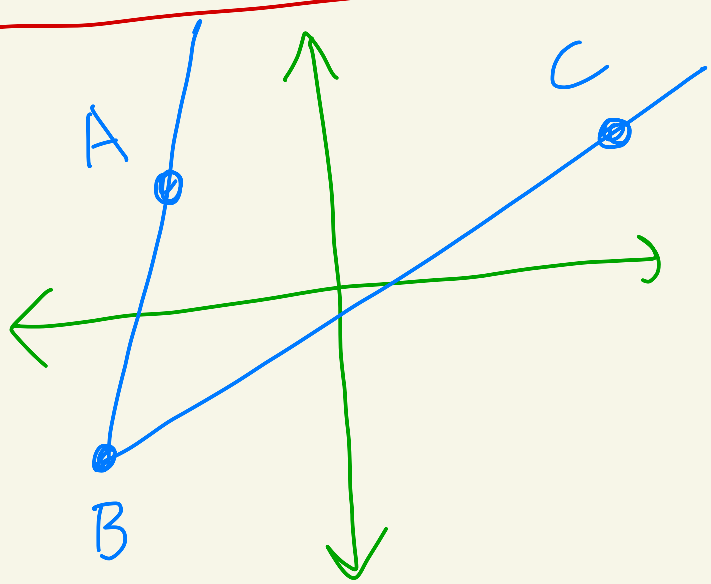
$\ln$ is 1-1, so $6 = \frac{2}{y}$.

So, $y = \frac{1}{3}$.

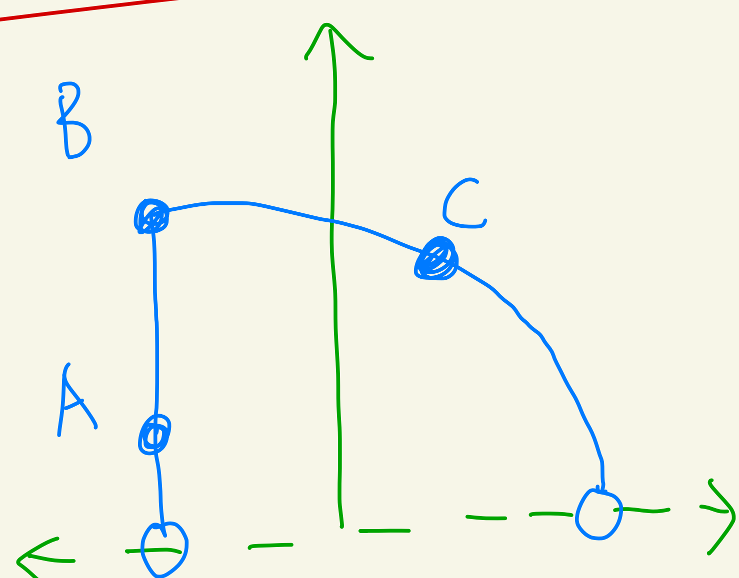
Thus, $\boxed{C = \left(0, \frac{1}{3}\right)}$.

Topic 6 - Angles and Triangles

Def: Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry. Let A, B, C be noncollinear points from $\mathscr{P}$.
The <u>angle</u> $\angle ABC$ is defined to be $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$



(angle in Euclidean plane)



(angle in hyperbolic plane)

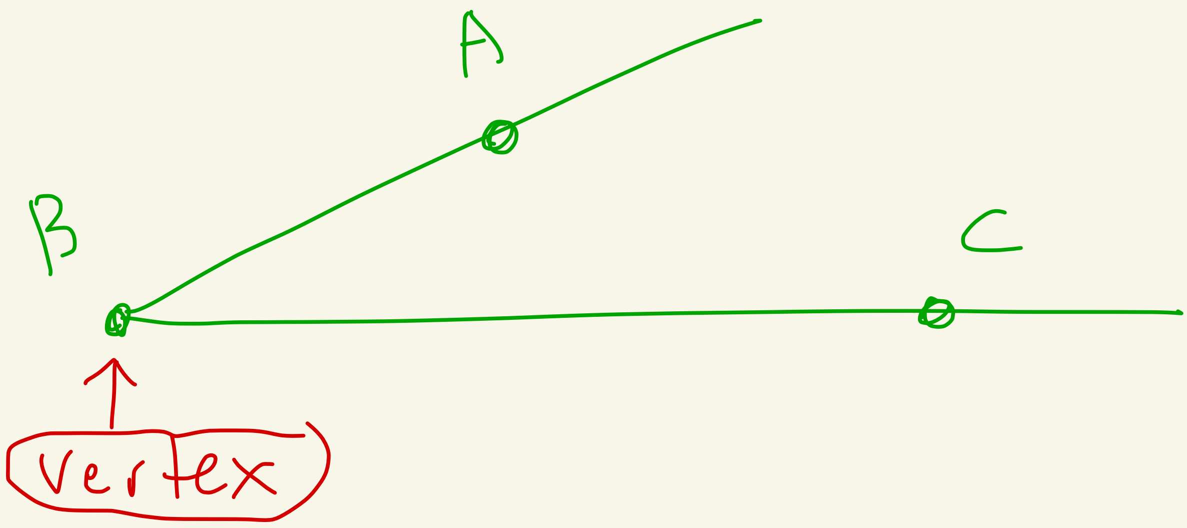
Note: A ray or line can't make an angle since the three points have to be noncollinear. That rules out "$0°$" and "$180°$" angles.

Theorem: Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry. Let A, B, C, D, E, F be points, where A, B, C are noncollinear and D, E, F are noncollinear. If $\angle ABC = \angle DEF$, then $B = E$.

Proof: See notes. $\blacksquare$

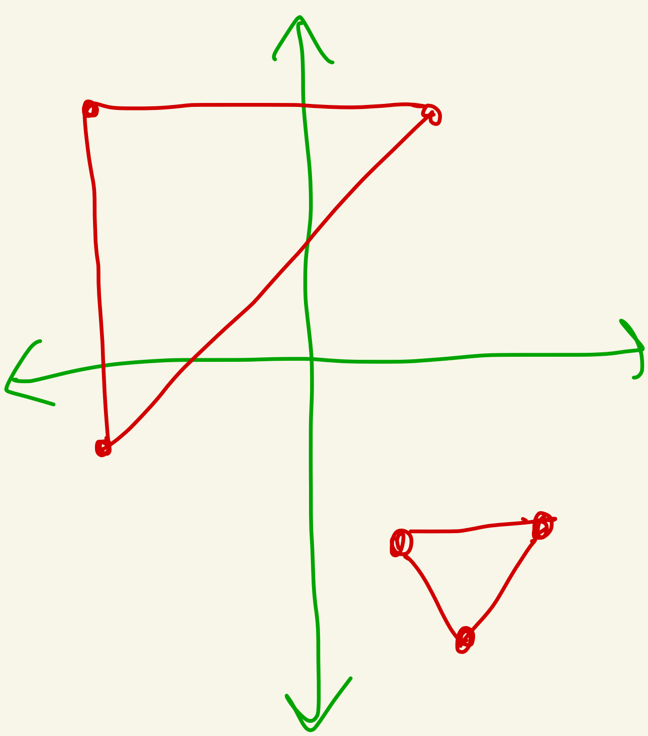
Thus, the following def is well-defined.

Def: Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry. Let A, B, C be noncollinear points. The **vertex** of $\angle ABC$ is B.

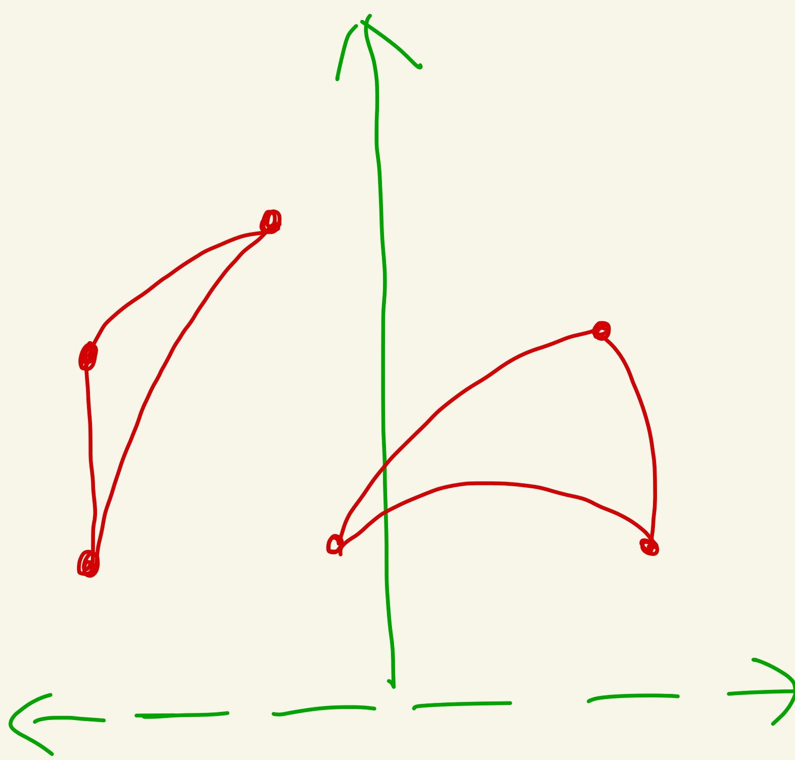


Def: Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry. Let A, B, C be noncollinear points. The ***triangle*** $\triangle ABC$ is defined to be

$$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$$



(some Euclidean triangles)



(some hyperbolic triangles)

Theorem: Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry. Let A, B, C be noncollinear points and D, E, F be noncollinear points.

If $\triangle ABC = \triangle DEF$, then $\{A, B, C\} = \{D, E, F\}$

proof: See notes. $\blacksquare$

This makes the next definition well-defined.

Def: Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry and A, B, C be noncollinear points.

The vertices of $\triangle ABC$ are A, B, C.

The sides of $\triangle ABC$ are $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$.

Note: By HW 6 #3(b),

$$\triangle ABC = \triangle BAC = \triangle CAB = \triangle ACB = \triangle BCA = \triangle CBA$$