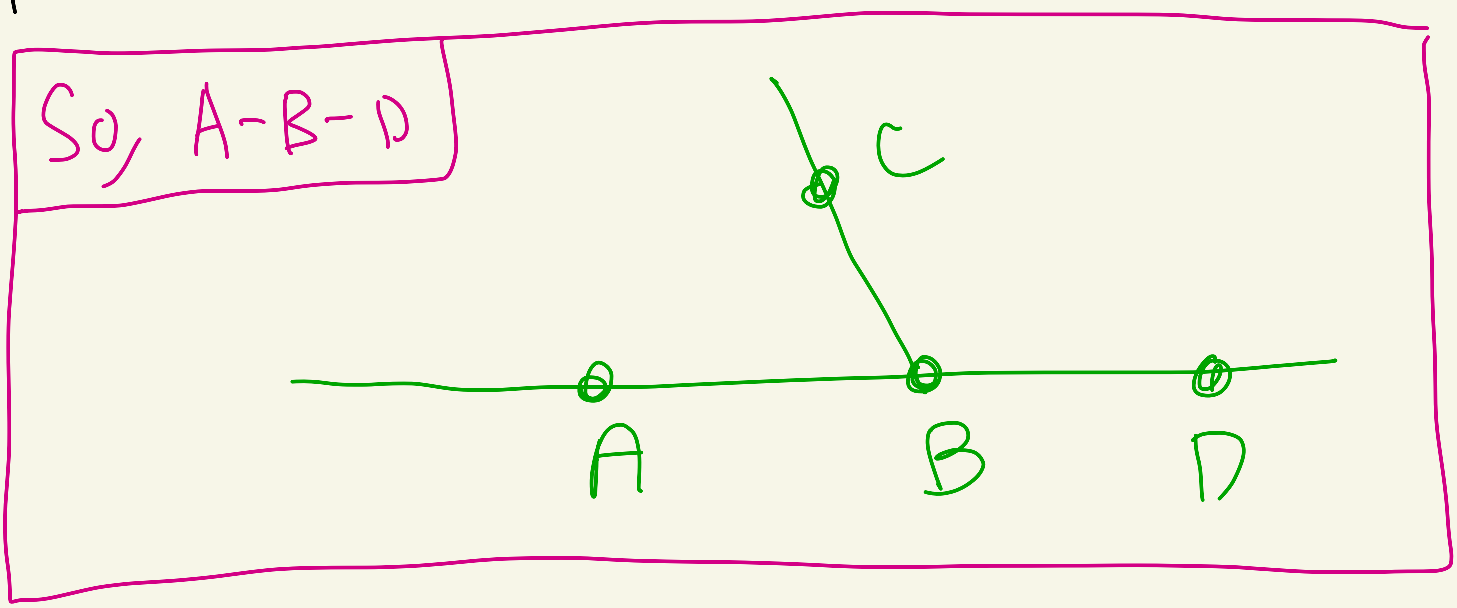
Math 4300

11/15/23

Theorem (Linear Pair theorem)

Suppose $\angle ABC$ and $\angle CBD$ form a linear pair in a protractor geometry $(\mathscr{P}, \mathscr{L}, d, m)$.

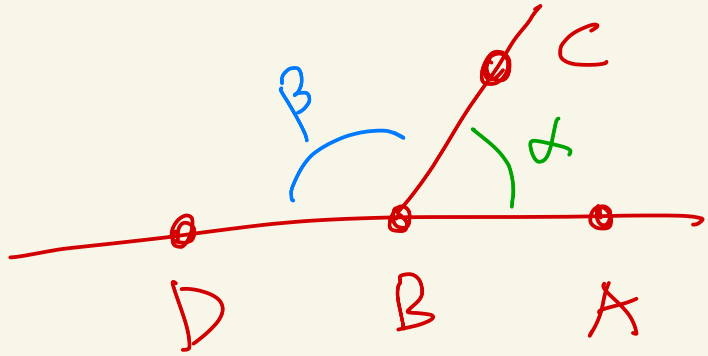


Then, $\angle ABC$ and $\angle CBD$ are supplementary, ie $m(\angle ABC) + m(\angle CBD) = 180$.

Proof: We are given that $A-B-D$.

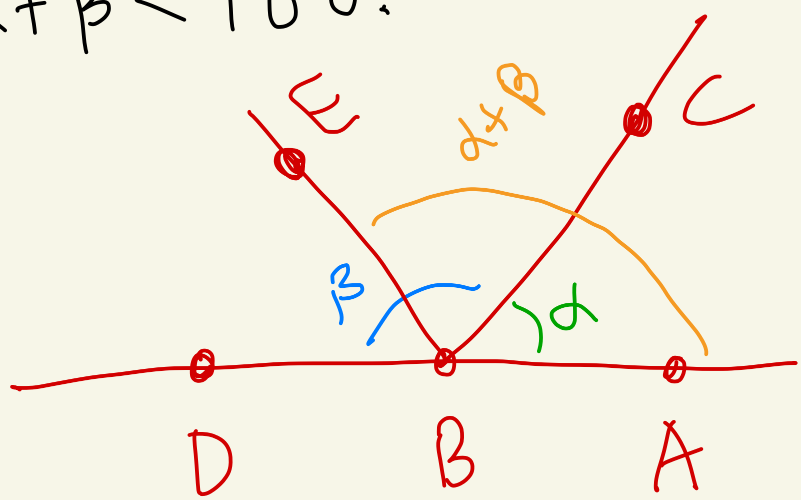
Let $\alpha = m(\angle ABC)$ and $\beta = m(\angle CBD)$

We must show that $\alpha + \beta = 180$.



To do this we show that $\alpha+\beta<180$ or $\alpha+\beta>180$ then we get a contradiction.

<u>case 1:</u> Suppose $\alpha+\beta<180$.



By property (iii) of m their exists a unique ray $\overrightarrow{BE}$ where

- E and C are on the same side of $\overleftrightarrow{AB}$
- $m(\angle ABE) = \alpha+\beta$

Since E and C are the same side of $\overleftrightarrow{AB}$ and

$m(\angle ABC) = \alpha < \alpha + \beta < m(\angle ABE)$

by lemma 1 we get that $C \in int(\angle ABE)$.

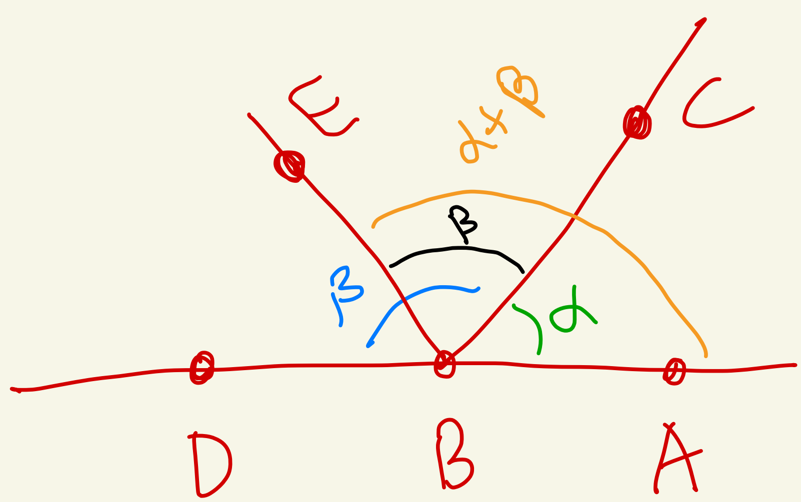
So by property (iii) of m we get

$m(\angle ABC) + m(\angle CBE) = m(\angle ABE)$

Thus, $\alpha + m(\angle CBE) = \alpha + \beta$

So, $m(\angle CBE) = \beta$.

Since $A-B-D$ and $C \in int(\angle ABE)$ by lemma 2 we get $E \in int(\angle CBD)$.



By property (iii) of m we get

$$m(\angle CBE) + m(\angle EBD) = m(\angle CBD)$$

Thus,

$$\beta + m(\angle EBD) = \beta$$

So, $m(\angle EBD) = 0$

This contradicts property (i) of m which says $0 < m(\angle EBD) < 180$.

So $\alpha + \beta < 0$ is impossible!

<u>Case 2:</u> Suppose $\boxed{\alpha + \beta > 180}$. $(*)$

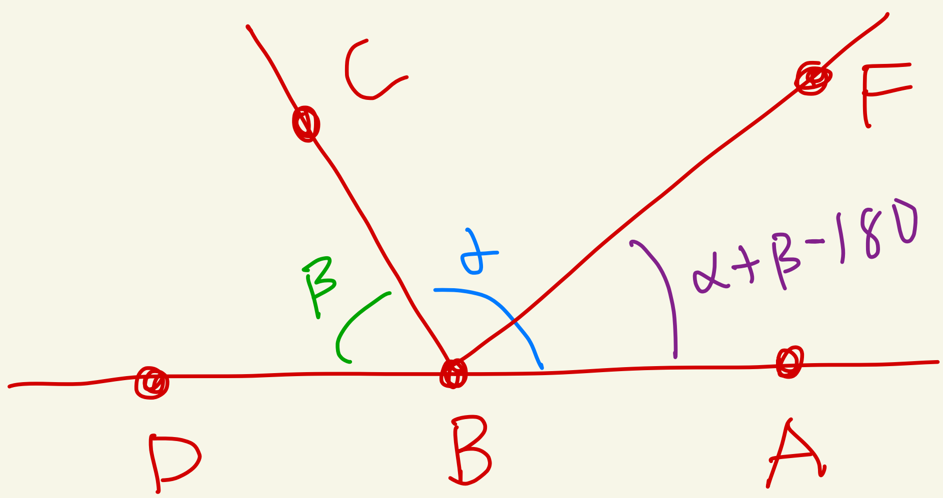
We know $0 < \alpha < 180$ and $0 < \beta < 180$ by property (i) of m.

So, $\boxed{0 < \alpha + \beta < 360}$. $(**)$

Thus, $0 \overset{(*)}{<} \alpha + \beta - 180 \overset{(**)}{<} 180$.

By property (ii) of m there is a unique ray $\overrightarrow{BF}$ with

- F and C are on the same side of $\overleftrightarrow{AB}$
- $m(\angle ABF) = \alpha + \beta - 180$.



Since $\beta < 180$ we know that

$$\alpha + \underbrace{\beta - 180}_{<0} < \alpha.$$

This says that $m(\angle ABF) < m(\angle ABC)$.

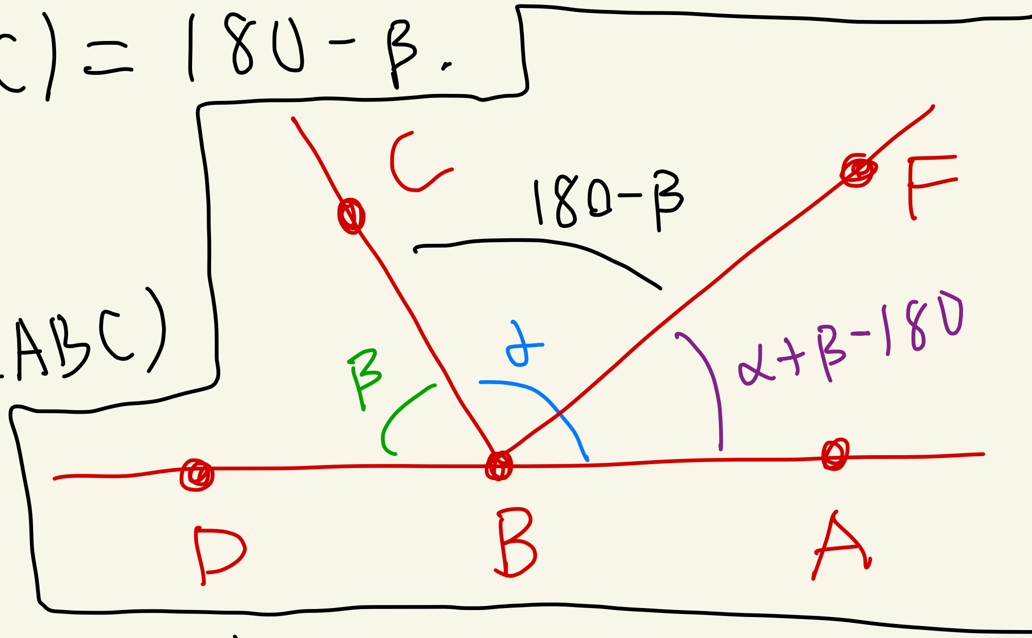
So, by lemma 1, we know $F \in \mathrm{int}(\angle ABC)$

Thus by property (iii) of m we get

$$m(\angle ABF) + m(\angle FBC) = m(\angle ABC)$$

So, $\alpha+\beta-180+m(\angle FBC)=\alpha$

Thus, $m(\angle FBC)=180-\beta$.



Since $A-B-D$ and $F\in int(\angle ABC)$ we know by lemma 2 that $C\in int(\angle FBD)$

Thus by property (iii) of m we get

$$m(\angle FBC)+m(\angle CBD)=m(\angle FBD).$$

So,

$$(180-\beta)+\beta=m(\angle FBD)$$

Thus,

$$m(\angle FBD)=180$$

which contradicts property 1 of m which says $0<m(\angle FBD)<180$.

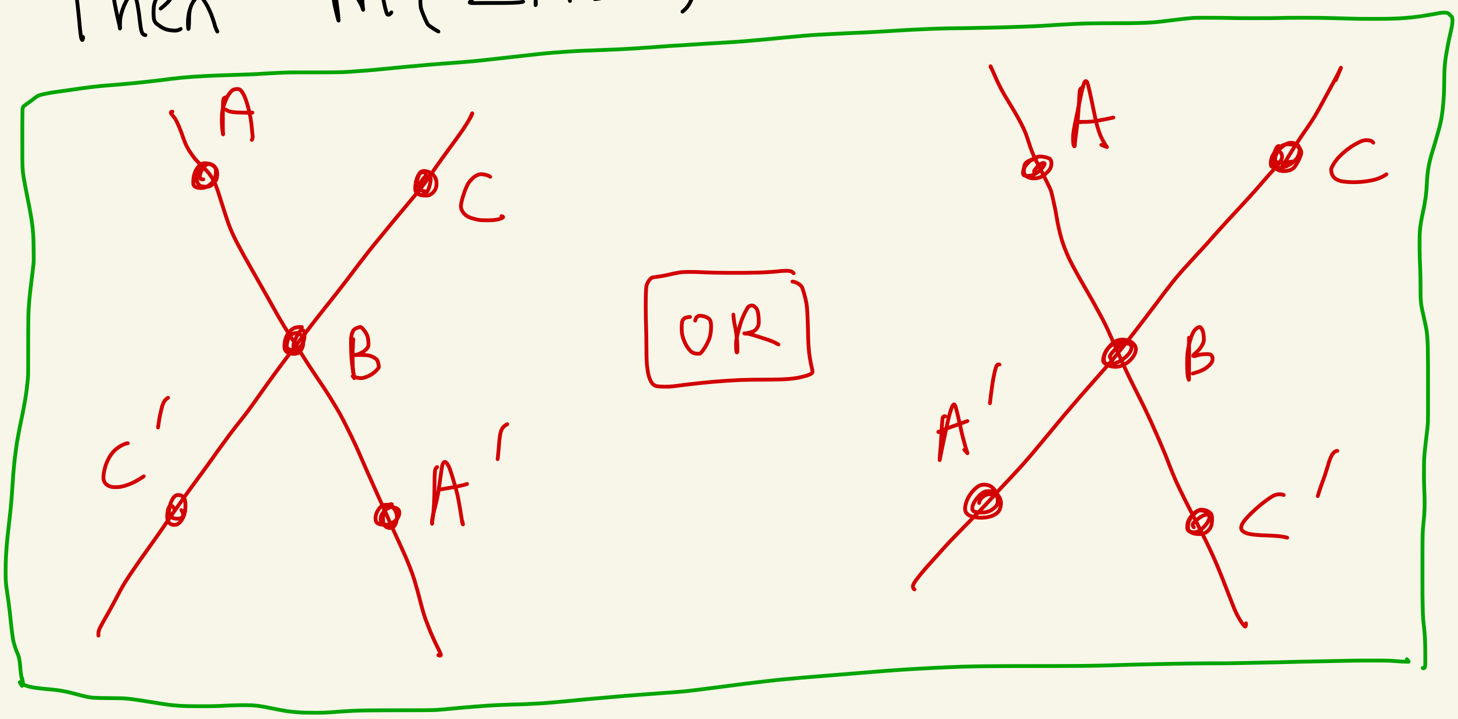
Therefore $\alpha + \beta > 180$.

Therefive, by cases 1/2 we know $\alpha + \beta = 180$. ∎

<u>Theorem (Vertical Angle Theorem)</u>

Let $\angle ABC$ and $\angle A'BC'$ be a vertical pair in a protractor geometry $(\mathcal{P}, \mathcal{L}, d, m)$.

Then $m(\angle ABC) = m(\angle A'BC')$



Proof: Since $\angle ABC$ and $\angle A'BC'$ form a vertical pair, either:

(i) $A-B-A'$ and $C-B-C'$

or (ii) $A-B-C'$ and $A'-B-C$

Suppose case (i) is true.

By the linear pair theorem we get:

$$\left.\begin{aligned}
&(a)\ m(\angle ABC)+m(\angle CBA')=180\\
&(b)\ m(\angle A'BC')+m(\angle C'BA)=180
\end{aligned}\right]\ \text{Since } A-B-A'$$

$$\left.\begin{aligned}
&(c)\ m(\angle C'BA)+m(\angle ABC)=180\\
&(d)\ m(\angle C'BA')+m(\angle A'BC)=180
\end{aligned}\right]\ \text{since } C'-B-C$$

Subtracting (d) from (a) gives:

$$\big(m(\angle ABC)+m(\angle CBA')\big)-\big(m(\angle C'BA')+m(\angle A'BC)\big)=0$$

Since $\angle CBA' = \angle A'BC$ we get

$$m(\angle ABC)-m(\angle C'BA')=0.$$

So, $m(\angle ABC) = m(A'BC')$.

Case (ii) is similar. $\blacksquare$

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11/27 Topic 12	11/29 Topic 12/13/14
12/4 Review	12/6 MIGHT BE CFA STRIKE
12/11 Final	