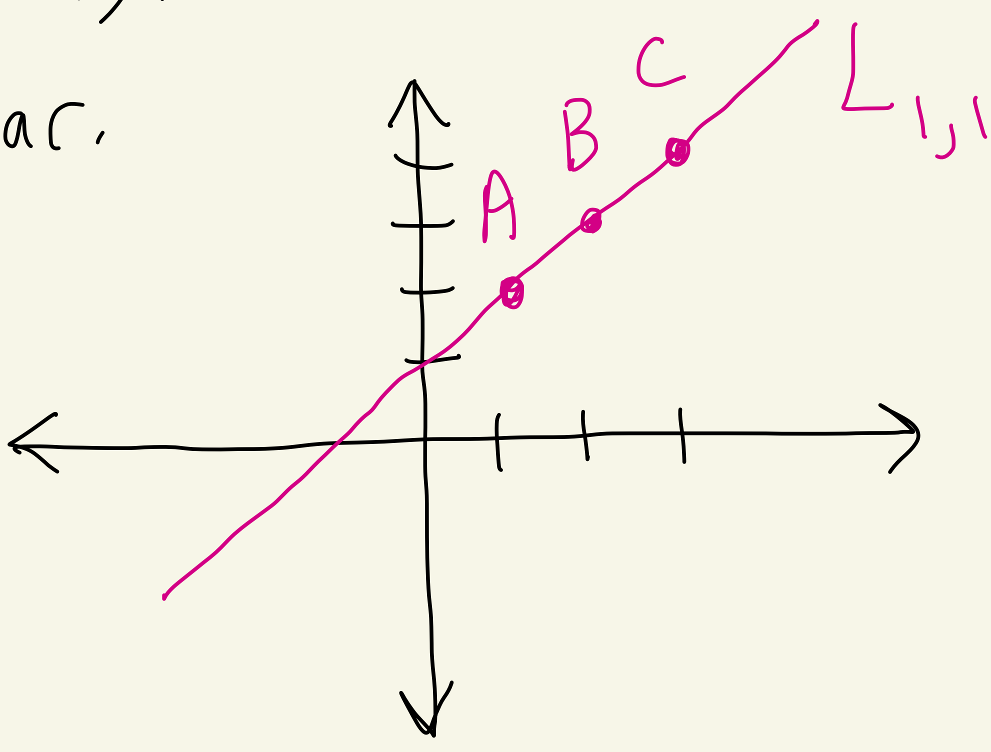
Math 4300
8/30/23

__Def:__ Let $(\mathscr{P}, \mathscr{L})$ be an abstract geometry. A set of points $S \subseteq \mathscr{P}$ is called <u>collinear</u> if there exists a line $\ell \in \mathscr{L}$ where $S \subseteq \ell$.

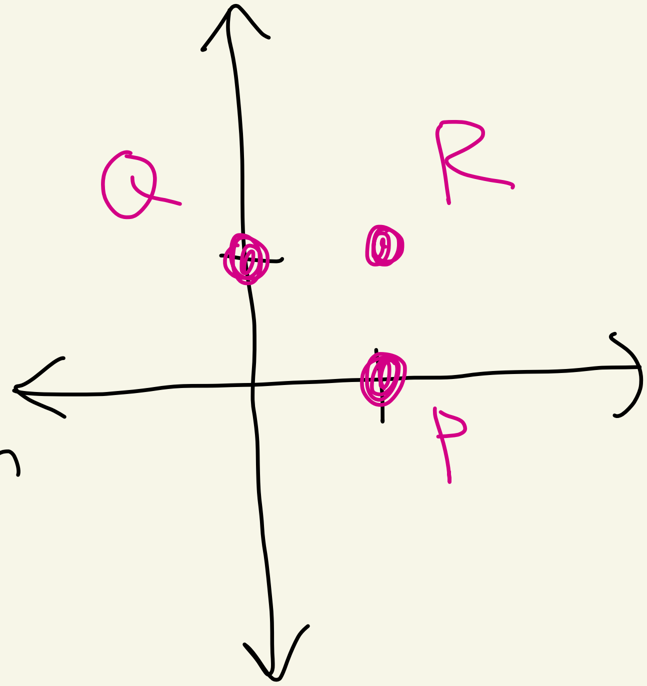
[That is, all of S lies on a line.]

If S is not collinear then we call the set S <u>non-collinear</u>.

__Ex:__ In the Euclidean plane $A=(1,2)$, $B=(2,3)$, $C=(3,4)$ are collinear.



However,
$P=(0,1)$, $Q=(1,0)$, $R=(1,1)$ are non-collinear.



There is no single line ℓ that goes through P, Q, and R.

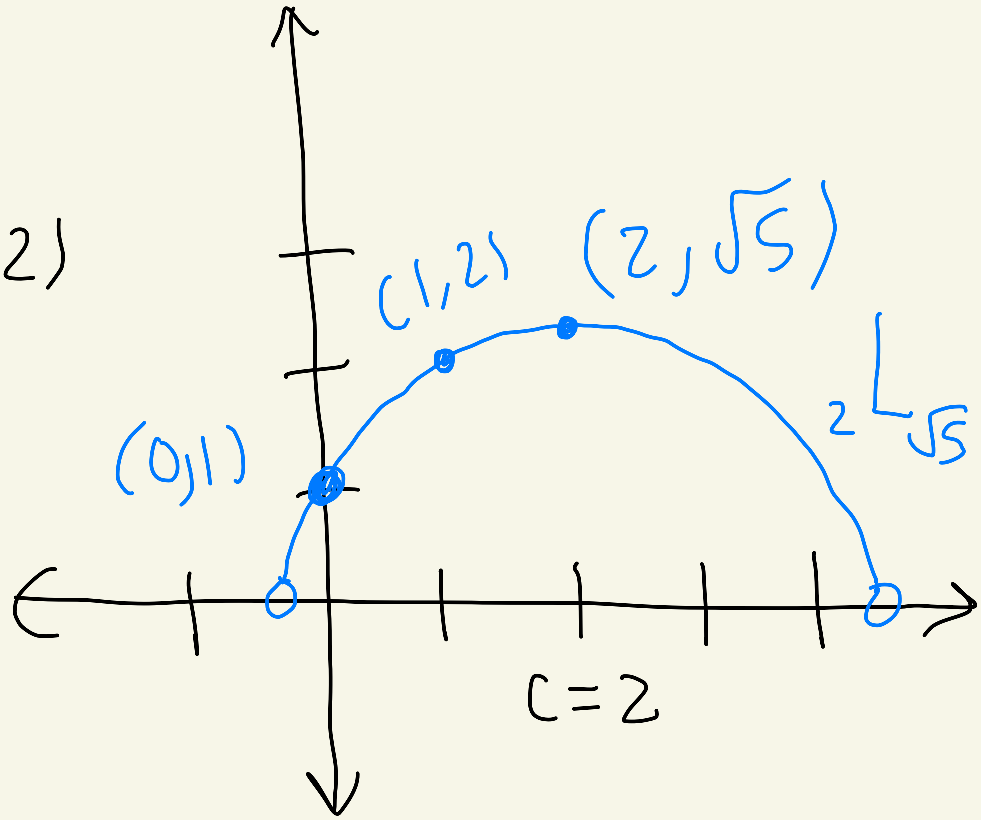
<u>Ex:</u> In the Hyperbolic plane the points $(0,1)$, $(1,2)$, $(2,\sqrt{5})$ are collinear, because they all lie on

$$ {}_2L_{\sqrt{5}} = \{(x,y)\in \mathbb{H} \mid (x-2)^2 + y^2 = \sqrt{5}^2\} $$

$\sqrt{5} \approx 2.236$

We already saw $(0,1), (1,2)$ lie on ${}_2L_{\sqrt{5}}$

And $(2,\sqrt{5}) \in {}_2L_{\sqrt{5}}$



because

$$(2-2)^2 + (\sqrt{5})^2 = (\sqrt{5})^2$$

$x = 2$ $\quad$ $y = \sqrt{5}$

Note that $(3,2), (4,1)$ also lie on ${}_2L_{\sqrt{5}}$.

So, $(0,1), (1,2), (2,\sqrt{5}), (3,2), (4,1)$ are all collinear.

Def: An abstract geometry $(\mathscr{P}, \mathscr{L})$ is called an <u>incidence geometry</u> if

(i) any two distinct points $P, Q \in \mathscr{P}$ lie on a unique line $l \in \mathscr{L}$.

(ii) there exist three points $A, B, C \in \mathscr{P}$ that are non-collinear.

<u>Note:</u> (i) adds "unique" to the abstract geometry.

(ii) says $\mathscr{P}$ is a "plane"

Notation: In an incidence geometry, the unique line ℓ that P and Q lie on is denoted by $\ell = \overleftrightarrow{PQ}$

Theorem: The Euclidean plane $\mathcal{E} = (\mathbb{R}^2, \mathcal{L}_E)$ is an incidence geometry.

proof: We already showed that $\mathcal{E}$ is an abstract geometry. Let's show that (i) and (ii) above hold.

Let's show (i).

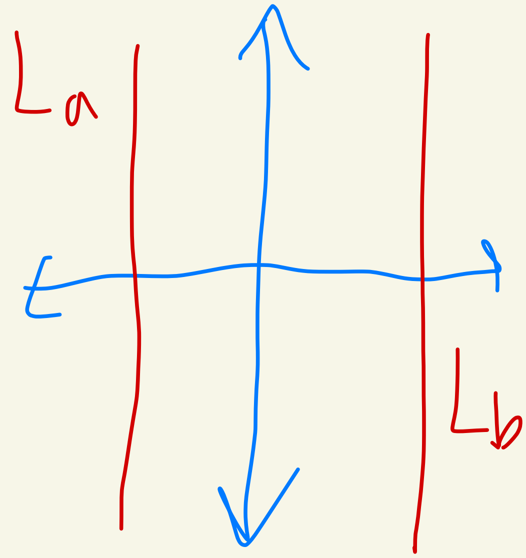
Let $P = (x_1, y_1)$, $Q = (x_2, y_2)$ be distinct points, that $P \neq Q$.

We already know, since $\mathcal{E}$ is an abstract geometry, that there exists a line through P and Q. We must show there is a <u>unique</u> line through P and Q.

<u>Case 1:</u> Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ lie on L_a and L_b where $a \neq b$.

Since $P, Q \in L_a$ we know

$a = x_1 = x_2$.



Since $P, Q \in L_b$ we know $b = x_1 = x_2$

But then $a = b$.

Contradiction. Can't happen.

<u>Case 2:</u> Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ both lie on L_a and $L_{m,b}$.

Since $P, Q \in L_a$ we know

$a = x_1 = x_2$.

Since $P, Q \in L_{m,b}$ we know

$$\underbrace{y_1 = m x_1 + b}_{P \in L_{m,b}} \quad \text{and} \quad \underbrace{y_2 = m x_2 + b}_{Q \in L_{m,b}}.$$

This implies that

$$y_1 = m x_1 + b = m a + b = m x_2 + b = y_2$$

$x_1 = a$ $\quad$ $x_2 = a$

But then

$$P = (x_1, y_1) = (x_2, y_2) = Q$$

Contradiction since $P \neq Q$.

<u>case 3:</u> Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ lie on $L_{m,b}$ and $L_{n,c}$ and

$L_{m,b} \neq L_{n,c}$.

Since $P, Q \in L_{m,b}$, we know

$\underbrace{y_1 = mx_1 + b}_{P \in L_{m,b}}$ and $\underbrace{y_2 = mx_2 + b}_{Q \in L_{m,b}}$

If $x_1 = x_2$, then $P, Q \in L_{x_1}$ which we dealt with in the previous case.

So we can assume $x_1 \neq x_2$.

Thus, $x_1 - x_2 \neq 0$.

Subtracting the eqns above gives

$$y_2 - y_1 = (mx_2 + b) - (mx_1 + b)$$
$$= m(x_2 - x_1)$$

Thus, $\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$. ← ok since $x_2 - x_1 \neq 0$

Since $y_1 = mx_1 + b$ we know

$$\boxed{b = y_1 - mx_1}.$$

Now do the same steps but use $L_{n,c}$ line and you'll get that

$\boxed{n = \frac{y_2 - y_1}{x_2 - x_1}}$ and $\boxed{c = y_1 - mx_1}$

Thus,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = n$$

and

$$b = y_1 - mx_1 = y_1 - mx_1 = c.$$

So, $L_{m,b} = L_{n,c}$.

Contradiction, since $L_{m,b} \neq L_{n,c}$.

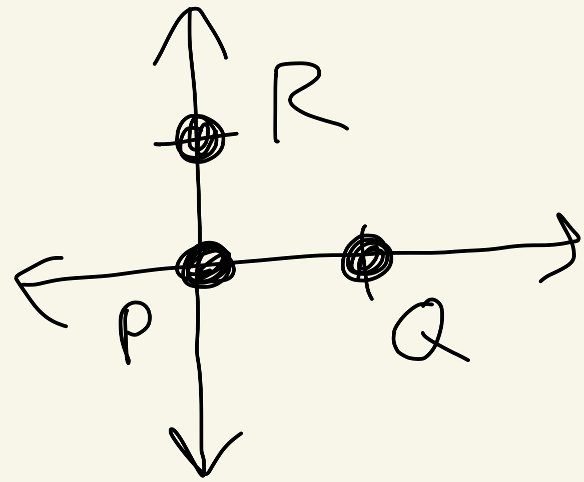
By cases 1, 2, 3 we have proven property (i).

Let's show property (ii)

We need three non-collinear

points in $\mathcal{E}$.

Consider $P=(0,0)$, $Q=(1,0)$ and $R=(0,1)$



Since these points don't all have the same x-coordinate they don't all lie on a vertical line.

Can we have $P, Q, R \in L_{m,b}$?

$y = mx + b$

If so, then

$0 = b \leftarrow P \in L_{m,b}$

$0 = m + b \leftarrow Q \in L_{m,b}$

$1 = b \leftarrow R \in L_{m,b}$

Can't happen!

Thus property (iii) is true.

So, $\mathcal{E}$ is an incidence geometry.

$\blacksquare$

<u>Theorem:</u> The hyperbolic plane $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$ is an incidence geometry.

<u>proof:</u> HW. $\blacksquare$

Theorem: Let $(\mathcal{P}, \mathcal{L})$ be an incidence geometry.
Let $l_1, l_2 \in \mathcal{L}$ be two lines.
If $l_1 \cap l_1$ contains two or more points, then $l_1 = l_2$.

proof: Suppose $P, Q \in l_1 \cap l_2$ where $P \neq Q$.
By prop (i) of incidence geometries there is a unique line through P and Q.

Since $P, Q \in l_1 \cap l_2$ we

Know $P, Q \in \ell_1$ and $P, Q \in \ell_2$.

Since $P, Q \in \ell_1$ we know

$$\ell_1 = \overleftrightarrow{PQ}.$$

Since $P, Q \in \ell_2$, we know

$$\ell_2 = \overleftrightarrow{PQ}.$$

So, $\ell_1 = \ell_2$. $\blacksquare$

Corollary: Let $(\mathcal{P}, \mathcal{L})$ be an incidence geometry. Let ℓ_1, ℓ_2 be two lines in $\mathcal{L}$.

Then either

(i) ℓ_1 and ℓ_2 are parallel

$[\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 = \phi]$

or

(ii) ℓ_1 and ℓ_2 intersect in exactly one point.

$[\ell_1 \cap \ell_2 = \{P\}]$

Proof: Let $\ell_1, \ell_2 \in \mathcal{L}$.

case 1: Suppose $\ell_1 \cap \ell_2 = \phi$.

Then, $l_1 \parallel l_2$.

<u>case 2:</u> Suppose $l_1 \cap l_2 = \{P\}$.
Then we are in (ii) above.

<u>case 3:</u> Suppose $l_1 \cap l_2$ has two or more points.
Then, the previous thm says that $l_1 = l_2$.

So, $l_1 \parallel l_2$.

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