Math 4300

9/18/23

Test 1 – moved to Weds 10/11

Was originally on 10/4

Theorem: Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry.
Let $l \in \mathscr{L}$ be a line and $f: l \to \mathbb{R}$ be a ruler/coordinate system for l.

Let $a \in \mathbb{R}$ and $\varepsilon = \pm 1$.
Define $h_{a,\varepsilon} : l \longrightarrow \mathbb{R}$ by

$$h_{a,\varepsilon}(P) = \varepsilon \underbrace{(f(P) - a)}_{\text{shifts } f \text{ by } a}$$

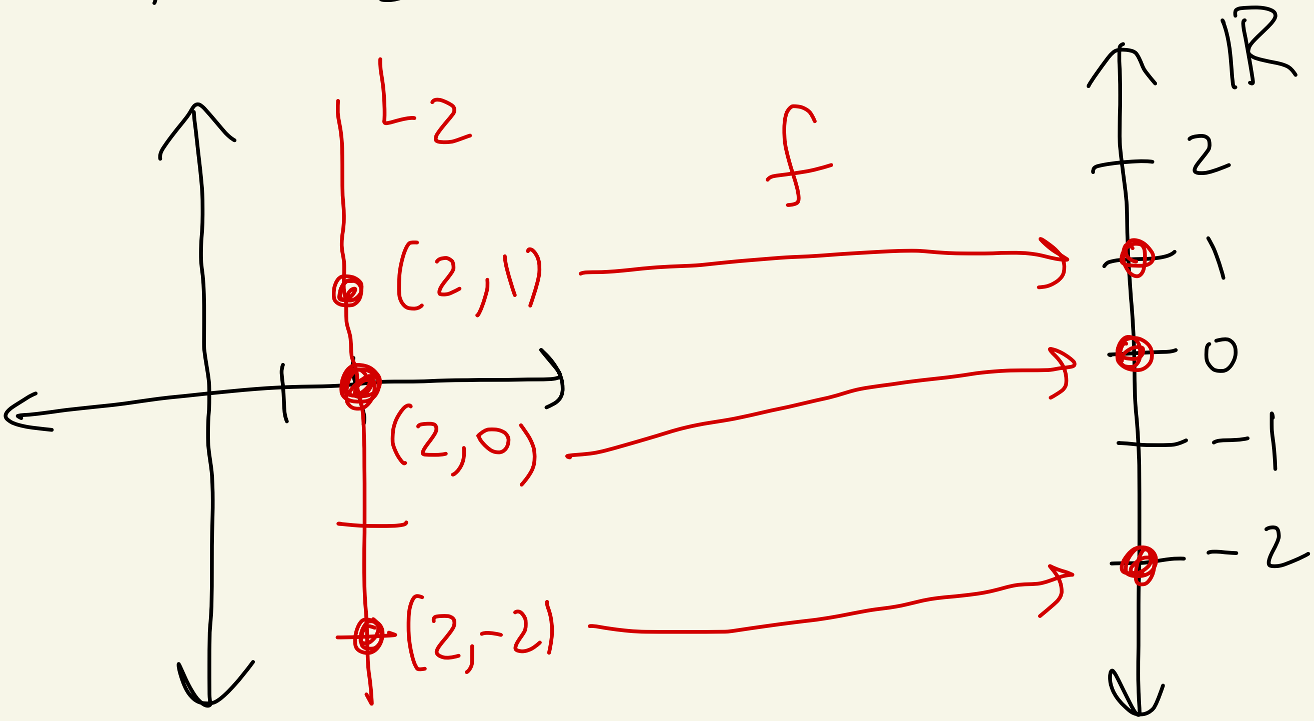
$\varepsilon = 1$ does nothing
$\varepsilon = -1$ flips f

Then, $h_{a,\varepsilon}$ will be a ruler for l.

Proof: See notes. $\blacksquare$

Ex: Consider the Euclidean plane. Let $\ell = L_2$.

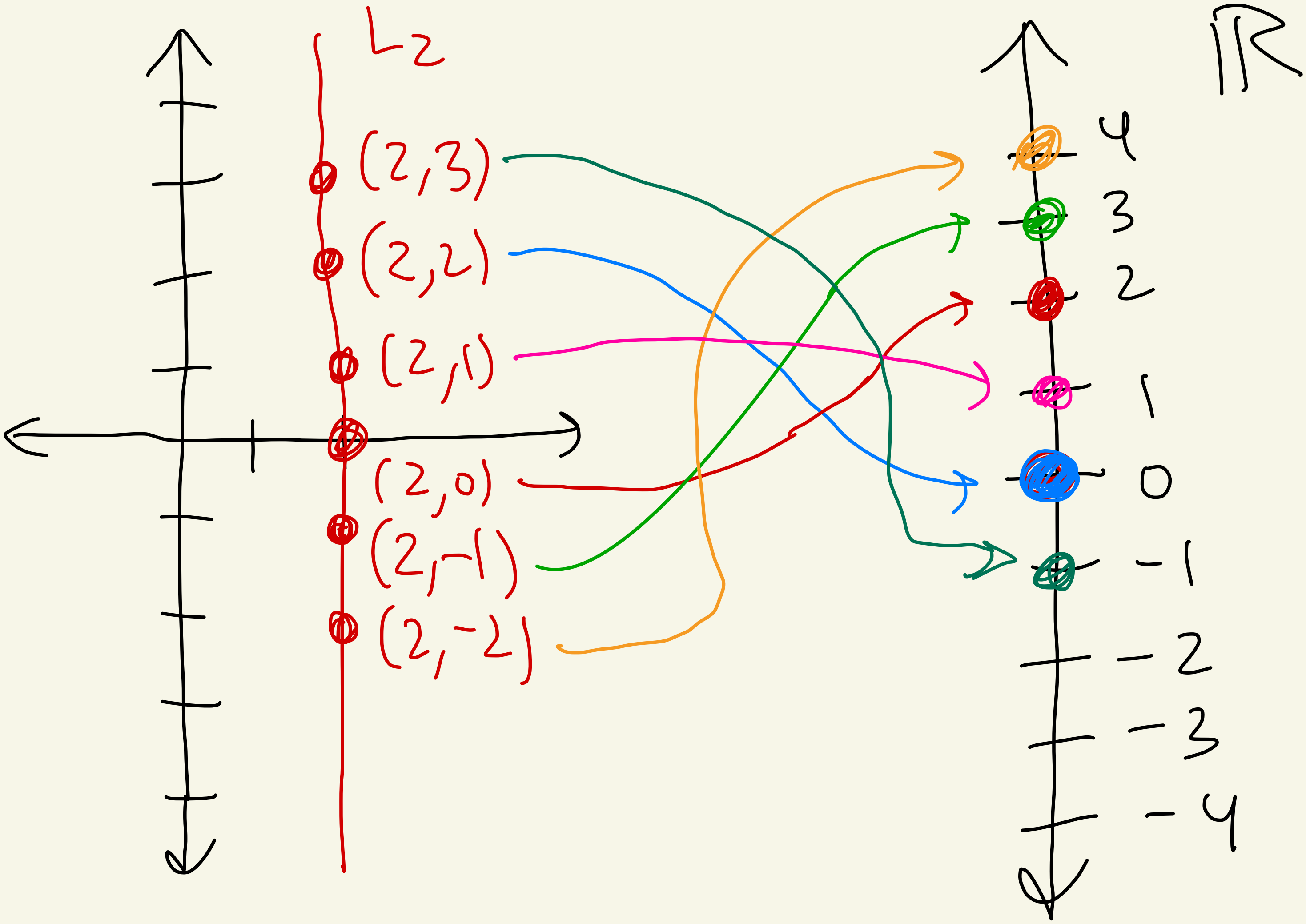
The standard ruler is $f : L_2 \to \mathbb{R}$ is $f(2,y) = y$



Let's make a new ruler using the above theorem with $a=2$ and $\varepsilon = -1$.

Let $h = h_{a,\varepsilon} = h_{2,-1}$ be as above.

So, $h(P) = (-1)\cdot\left[f(P) - 2\right]$



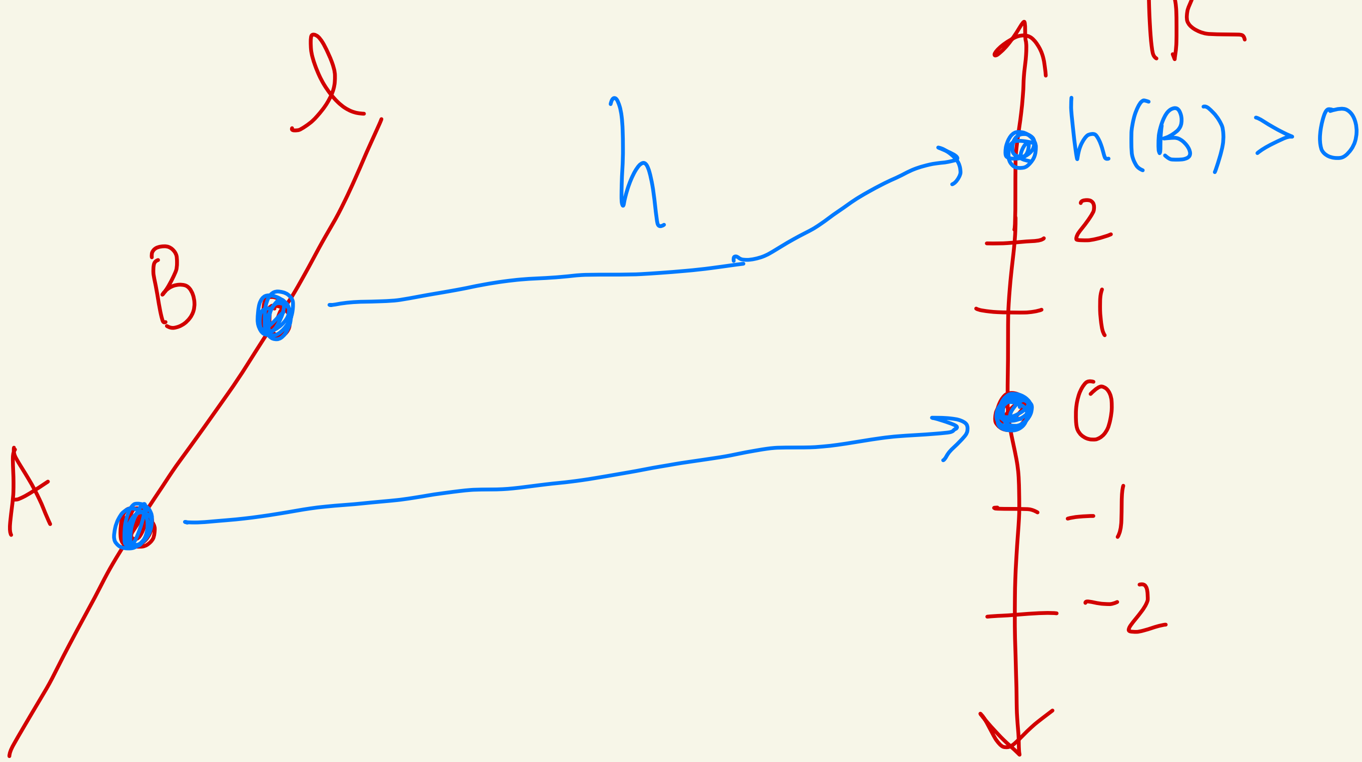
$$h(2,0) = -[f(2,0) - 2] = -[0-2] = 2$$

$$h(2,2) = -[f(2,2) - 2] = -[2-2] = 0$$

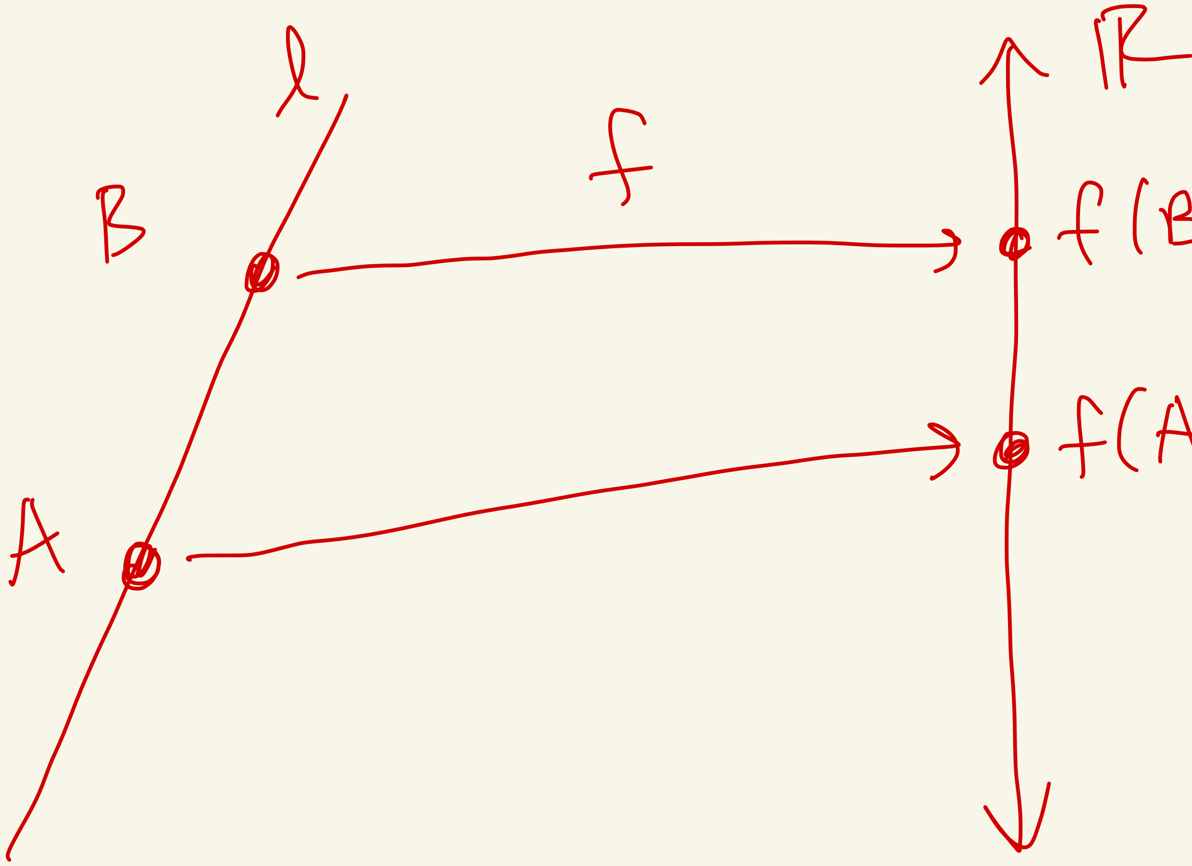
Theorem: (Ruler placement theorem)

Let $(\mathcal{P}, \mathcal{L}, d)$ be a metric geometry. Let $\ell \in \mathcal{L}$ be a line. Let $A, B \in \ell$.

Then there exists a ruler $h : \ell \to \mathbb{R}$ where $h(A) = 0$ and $h(B) > 0$.



__proof:__ We are in a metric geometry. Thus there exists a ruler $f:\ell\to\mathbb{R}$ for ℓ.



$$\boxed{\text{Case 1: Suppose } f(B) > f(A)}$$

Then define

$$h(P) = \underbrace{1}_{\boxed{\varepsilon=1}}\cdot\underbrace{\big(f(P)-f(A)\big)}_{\text{shift } f \text{ by } f(A)}$$

Then h is a ruler by the previous theorem and

$$h(A) = f(A) - f(A) = 0$$

$$h(B) = f(B) - f(A) > 0$$

$\boxed{f(B) > f(A)}$

$\boxed{\text{Case 2: Suppose } f(B) < f(A)}$

Then define

$$h(P) = \underbrace{(-1)}_{\boxed{\varepsilon = -1}} \cdot \Big[\underbrace{f(P) - f(A)}_{\text{shift } f \text{ by } f(A)}\Big]$$

Then h is a ruler by the previous theorem and

$$h(A) = (-1) \cdot [f(A) - f(A)] = 0$$

$$h(B) = (-1) \cdot [f(B) - f(A)]$$

$$= f(A) - f(B) > 0$$

since $f(B) < f(A)$

Example: Consider the Euclidean plane. Let $A = (1,0)$ and $B = (-1,2)$.

Let $\ell = \overleftrightarrow{AB}$

Find a ruler $h : \ell \to \mathbb{R}$ where $h(A) = 0$ and $h(B) > 0$.

Solution:

A and B lie on

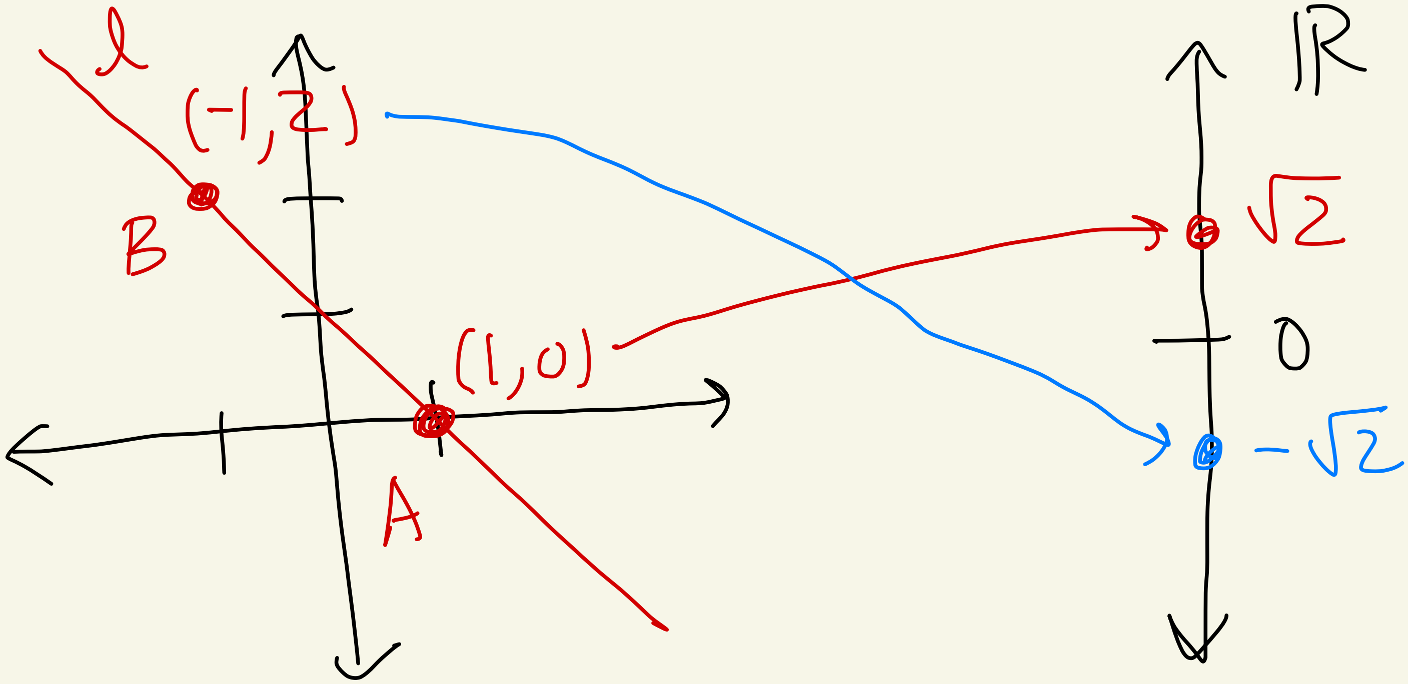
$$\ell = L_{-1,1} = \{ (x,y) \mid y = -x + 1 \}$$

The standard is $f : \ell \to \mathbb{R}$ where $f(x,y) = \sqrt{2}\, x$

$m = -1$

standard ruler on $L_{m,b}$

$f(x,y) = x\sqrt{1+m^2}$



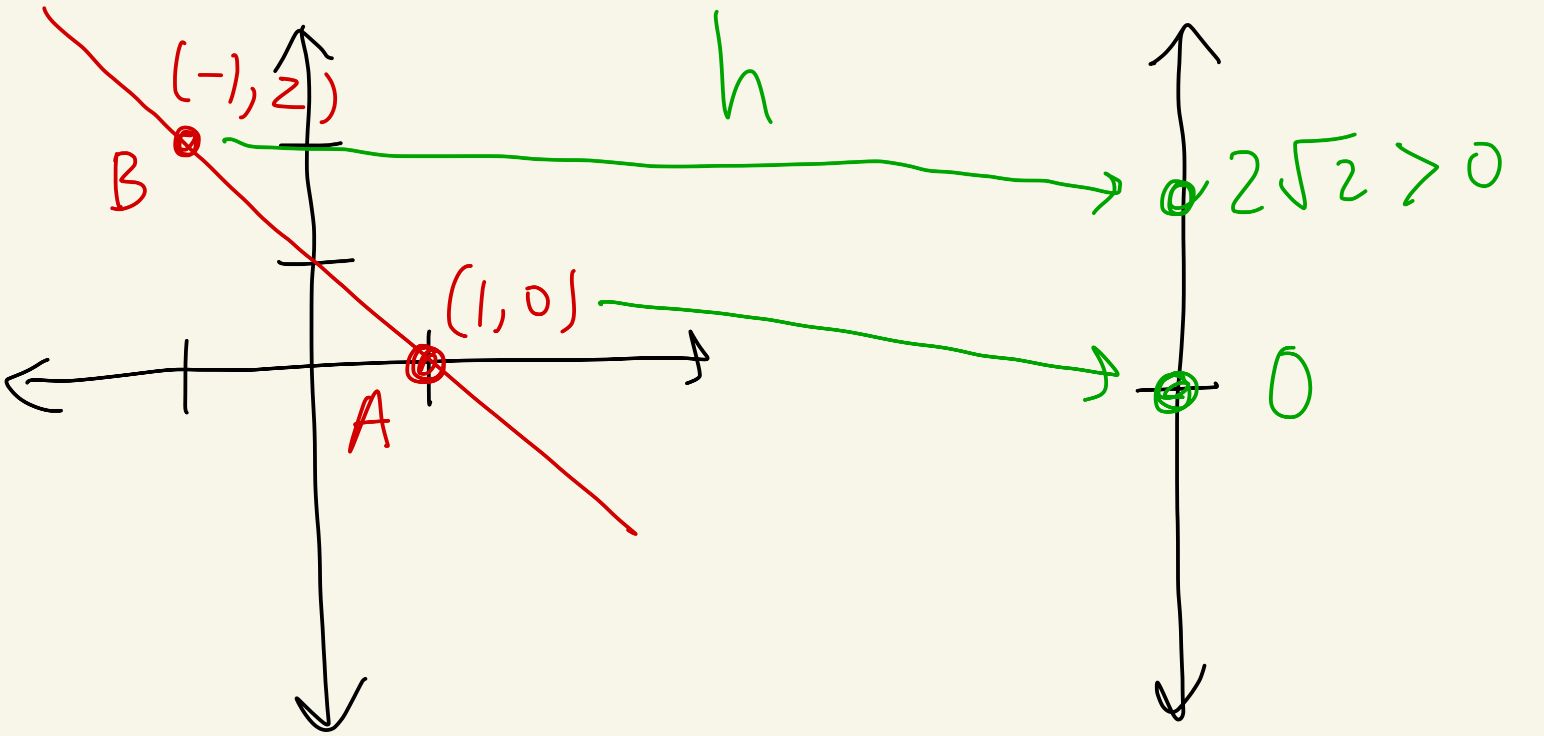
In this case $f(B) < f(A)$
so we need to both shift
and flip!

The formula for h is

$$h(P) = \underbrace{(-1)}_{\substack{\text{flip} \\ \varepsilon = -1 \\ \text{to make} \\ h(B) > 0}} \cdot \underbrace{\left[f(P) - f(A)\right]}_{\substack{\text{Shift to make} \\ h(A) = 0}}$$

$$= f(A) - f(P)$$

So, $h(x,y) = f(A) - f(x,y)$
$= \sqrt{2} - \sqrt{2}x$] $P = (x,y)$



$h(A) = h(1,0) = \sqrt{2} - \sqrt{2}(1) = 0$

$h(B) = h(-1,2) = \sqrt{2} - \sqrt{2}(-1) = 2\sqrt{2} > 0$

Theorem: Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry. Let $\ell \in \mathscr{L}$ and $f : \ell \to \mathbb{R}$ be a ruler. If $g : \ell \to \mathbb{R}$ is another ruler for ℓ, then there exists $a \in \mathbb{R}$ and $\varepsilon = \pm 1$ where

$$g(P) = \varepsilon\big(f(P) - a\big)$$

for all $P \in \ell$.

proof: See Millman/Parker page 39. $\blacksquare$

$\boxed{\text{Topic 3 - More on the Euclidean plane}}$

<u>Def:</u> Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be in $\mathbb{R}^2$.

Let $\alpha \in \mathbb{R}$.

Define the following:

(i) $A + B = (x_1 + x_2, y_1 + y_2)$

(ii) $\alpha A = (\alpha x_1, \alpha y_1)$

(iii) $A - B = (x_1 - x_2, y_1 - y_2)$

(iv) $\langle A, B \rangle = x_1 x_2 + y_1 y_2$

(v) $\|A\| = \sqrt{\langle A, A \rangle}$

$$= \sqrt{x_1^2 + y_1^2}$$

<u>Ex:</u> Let $A = (1,2)$ and $B = (-1,5)$

Then,

$$A + B = (1-1,\ 2+5) = (0,7)$$

$$3A = (3(1),\ 3(2)) = (3,6)$$

$$A - B = (1-(-1),\ 2-5) = (2,-3)$$

$$\langle A, B \rangle = (1)(-1) + (2)(5) = 9$$

$$\|A\| = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$$