Math 4300

9/20/23

Theorem: Let $A, B \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$. Then:

(i) $A + B = B + A$

(ii) $A + (B + C) = (A + B) + C$

(iii) $\alpha(A + B) = \alpha A + \alpha B$

(iv) $(\alpha + \beta) A = \alpha A + \beta A$

(v) $\langle A, B \rangle = \langle B, A \rangle$

(vi) $\langle \alpha A, B \rangle = \alpha \langle A, B \rangle$

(vii) $\langle A + B, C \rangle = \langle A, C \rangle + \langle B, C \rangle$

(viii) $\|\alpha A\| = |\alpha| \cdot \|A\|$

(ix) $\|A\| > 0$ iff $A \neq (0,0)$

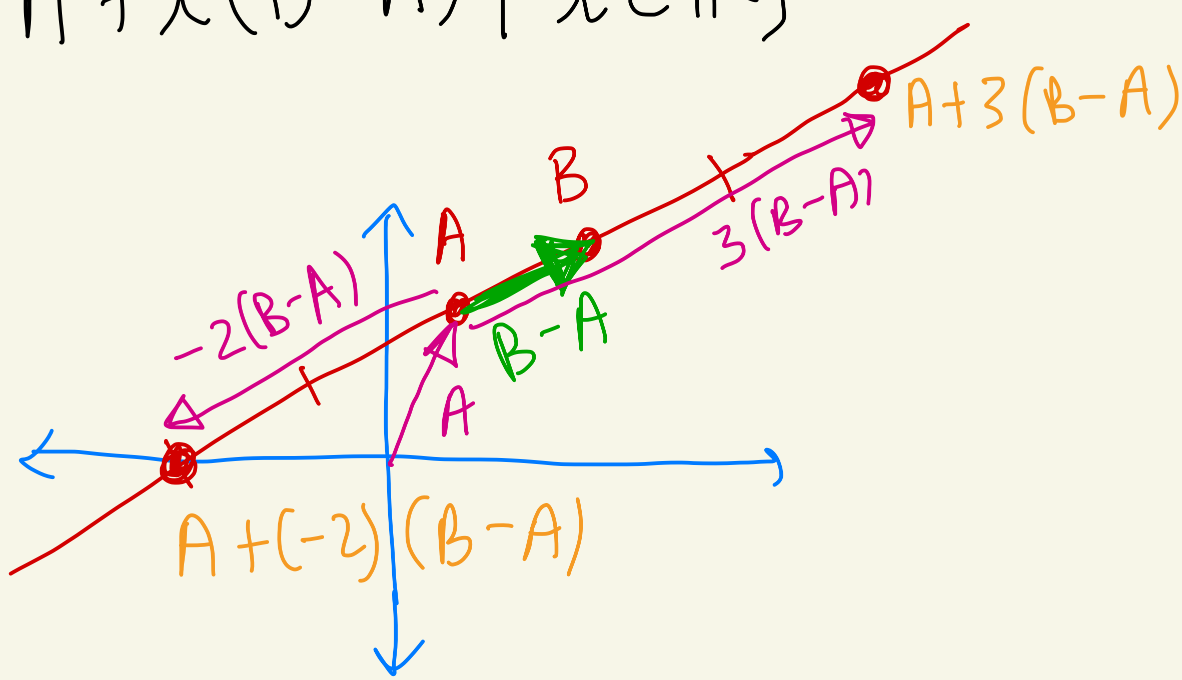
[same as: $\|A\| = 0$ iff $A = (0,0)$]

proof: HW 3 $\blacksquare$

We are now going to re-describe lines in the Euclidean plane using the vector equation of a line from Calculus.

$\boxed{\text{Def:}}$ Let A and B be two distinct points from $\mathbb{R}^2$.
Define

$$L_{AB} = \{ A + t(B-A) \mid t \in \mathbb{R} \}$$



In calculus, the vector eqn of the line through A and B is

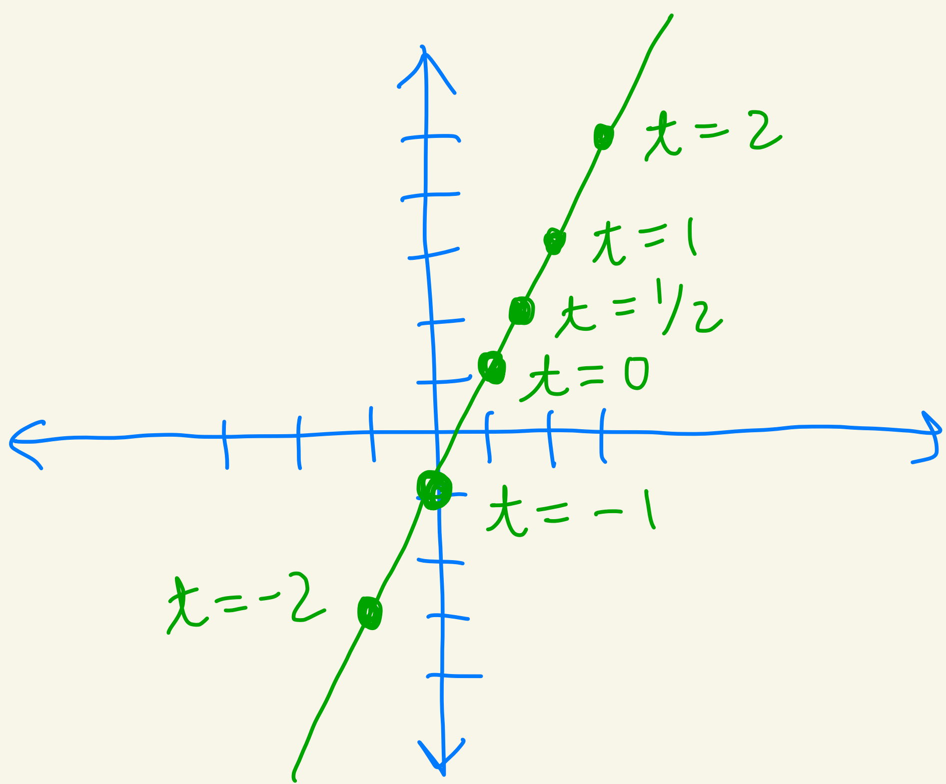
$$\boxed{A + t(B-A)}$$

<u>Ex:</u> $A = (1,1)$, $B = (2,3)$

$$L_{AB} = \{ \underbrace{(1,1)}_{A} + t \underbrace{(1,2)}_{B-A} \mid t \in \mathbb{R} \}$$

$$= \{ (1+t, 1+2t) \mid t \in \mathbb{R} \}$$

t	$(1+t, 1+2t)$
0	(1,1)
1	(2,3)
-1	(0,-1)
2	(3,5)



-2	$(-1,-3)$
$1/2$	$(3/2, 2)$
$\vdots$	$\vdots$

<u>Theorem:</u> Let

$$\mathcal{L}' = \{ L_{AB} \mid A \text{ and } B \text{ are distinct pts in } \mathbb{R}^2 \}$$

Then,

(i) $\mathcal{L}' = \mathcal{L}_E$

$\mathcal{L}_E$ was all the Euclidean lines $L_{m,b}$ and L_a

(ii) L_{AB} is the unique line through A and B

<u>Proof:</u> See the notes I email to you $\blacksquare$

Theorem: If $A, B \in \mathbb{R}^2$, then

$$\underbrace{d_E(A,B)}_{\text{Euclidean distance}} = \|A-B\|$$

proof: HW 3. $\blacksquare$

Ex: $A=(1,1)$ and $B=(2,3)$

$$d_E(A,B) = \sqrt{(1-2)^2+(1-3)^2}$$

$$= \sqrt{1+4} = \sqrt{5}$$

and

$$\|A-B\| = \|(-1,-2)\|$$
$$= \sqrt{(-1)^2+(-2)^2} = \sqrt{5}$$

<u>Theorem:</u> Consider the Euclidean metric geometry

$$\mathcal{E} = (\mathbb{R}^2, \underbrace{\mathcal{L}_E}_{\text{same as } \mathcal{L}' \text{ above}}, d_E).$$

Let A, B be distinct points.
Let L_{AB} be the line through A and B.
Then, $f: L_{AB} \to \mathbb{R}$ defined

by $f(\underbrace{A+t(B-A)}_{\text{point on line } L_{AB}}) = t\|B-A\|$

is a ruler for L_{AB}.

$\boxed{\text{Note:}}$ $f(A) = f(A+0(B-A)) = 0\|B-A\|$
$= 0$

and $f(B) = f(A+1\cdot(B-A)) = 1\cdot\|B-A\|$
> 0

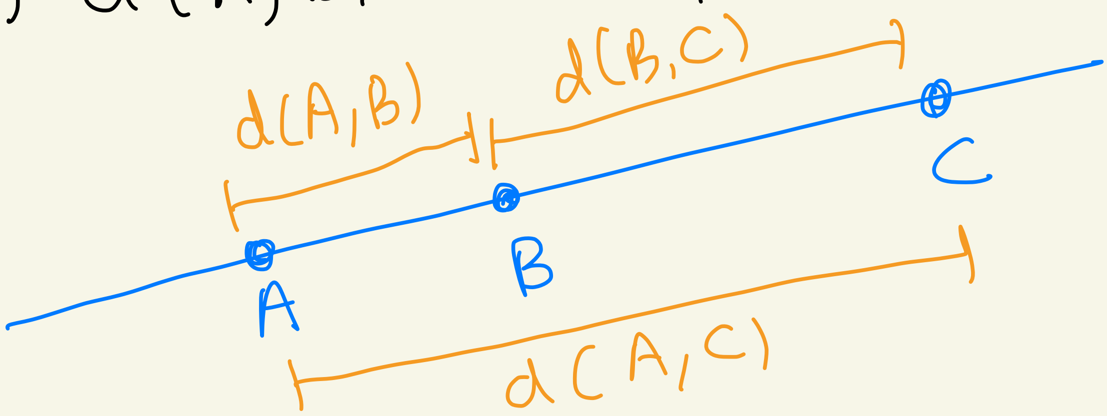
So, $f(A)=0$ and $f(B)>0$.

<ins>proof:</ins> See notes. $\blacksquare$

Topic 4 – Betweenness

<u>Def:</u> Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry.
Let $A, B, C \in \mathscr{P}$ be points.
We say that <u>B is between A and C</u> if

(i) A, B, C are distinct points
(ii) A, B, C are collinear
(iii) $d(A,B) + d(B,C) = d(A,C)$



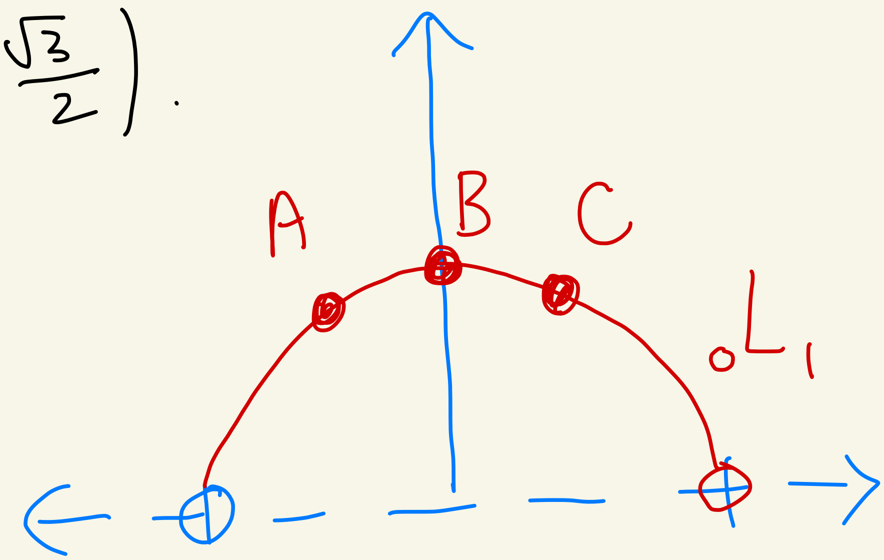
We write $A - B - C$ to mean

that B is between A and C.

Ex: Consider the hyperbolic plane $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H, d_H)$

Let $A = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $B = (0,1)$, and $C = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$\boxed{\text{Claim: } A - B - C}$



(i) A, B, C are distinct points.

(ii) A, B, C are collinear since they all lie on

$${}_0L_1 = \{ (x,y) \in \mathbb{H} \mid \underbrace{(x-0)^2 + y^2 = 1}_{x^2+y^2=1} \}$$

(iii) Recall that on ${}_0L_1$ the distance function is $\boxed{\begin{matrix} c=0 \\ r=1 \end{matrix}}$

$$d_H((x_1,y_1),(x_2,y_2)) = \left| \ln\left(\frac{\frac{x_1-c+r}{y_1}}{\frac{x_2-c+r}{y_2}} \right) \right|$$

$$= \left| \ln\left(\frac{\frac{x_1+1}{y_1}}{\frac{x_2+1}{y_2}} \right) \right|$$

So, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $(0,1)$

$$d_H(A,B) = \left| \ln\left(\frac{\frac{-1/2+1}{\sqrt{3}/2}}{\frac{0+1}{1}} \right) \right| = \left| \underbrace{\ln\left(\frac{1}{\sqrt{3}}\right)}_{\text{negative}} \right|$$

$$= -\ln\left(\frac{1}{\sqrt{3}}\right)$$

$\boxed{-\ln(t) = \ln(t^{-1})}$ $\quad = \ln(\sqrt{3})$

$$d_H(\underset{(0,1)}{B},\underset{\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)}{C}) = \left|\ln\left(\frac{\frac{0+1}{1}}{\frac{1/2+1}{\sqrt{3}/2}}\right)\right| = \left|\underbrace{\ln\left(\frac{\sqrt{3}}{3}\right)}_{\text{negative}}\right|$$

$$= -\ln\left(\frac{\sqrt{3}}{3}\right) = \ln\left(\frac{3}{\sqrt{3}}\right)$$

$$d_H(\underset{(-1/2,\sqrt{3}/2)}{A},\underset{\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)}{C}) = \left|\ln\left(\frac{\frac{-1/2+1}{\sqrt{3}/2}}{\frac{1/2+1}{\sqrt{3}/2}}\right)\right| = \left|\ln\left(\frac{1/\sqrt{3}}{3/\sqrt{3}}\right)\right|$$

$$= \left|\underbrace{\ln\left(\frac{1}{3}\right)}_{\text{negative}}\right| = -\ln\left(\frac{1}{3}\right) = \ln(3)$$

So, $d_H(A,B) + d_H(B,C)$

$$= \ln(\sqrt{3}) + \ln\left(\frac{3}{\sqrt{3}}\right)$$

$$\ln(x)+\ln(y) = \ln(xy)$$

$$= \ln\left(\sqrt{3}\cdot\frac{3}{\sqrt{3}}\right) = \ln(3) = d_H(A,C)$$

$\boxed{\text{claim}}$