Math 4300

9/25/23

Notation:

We will write AB for $d(A,B)$

Theorem: Let $(\mathcal{P}, \mathcal{L}, d)$ be a metric geometry with $A, B, C \in \mathcal{P}$.

If $A-B-C$, then $C-B-A$.

Proof:

Suppose $A-B-C$.

Then,

① A, B, C are distinct

② A, B, C are collinear

(3) $\underbrace{AC}_{d(A,C)} = \underbrace{AB}_{d(A,B)} + \underbrace{BC}_{d(B,C)}$

$\boxed{A-B-C}$

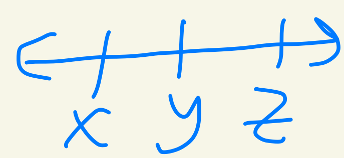
$XY = YX$

$d(x,y) = d(y,x)$

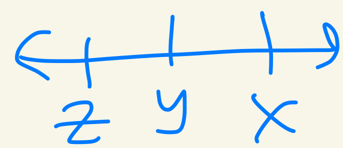
So, $CA = CB + BA$.

Thus, $C-B-A$.

Def: If $x, y, z \in \mathbb{R}$, we say that *y is between x and z* if $x < y < z$

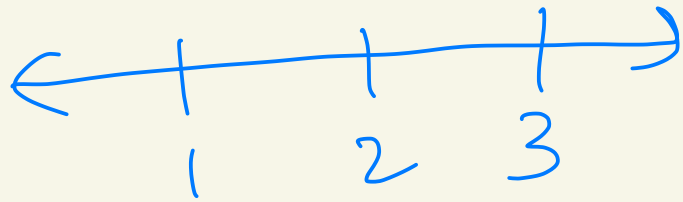


or $z < y < x$.



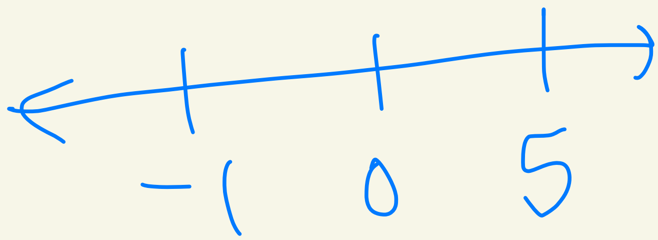
We denote this by $x * y * z$ to mean either $x<y<z$ or $z<y<x$.

<u>Ex:</u> $1 * 2 * 3$



$1<2<3$

$5 * 0 * -1$



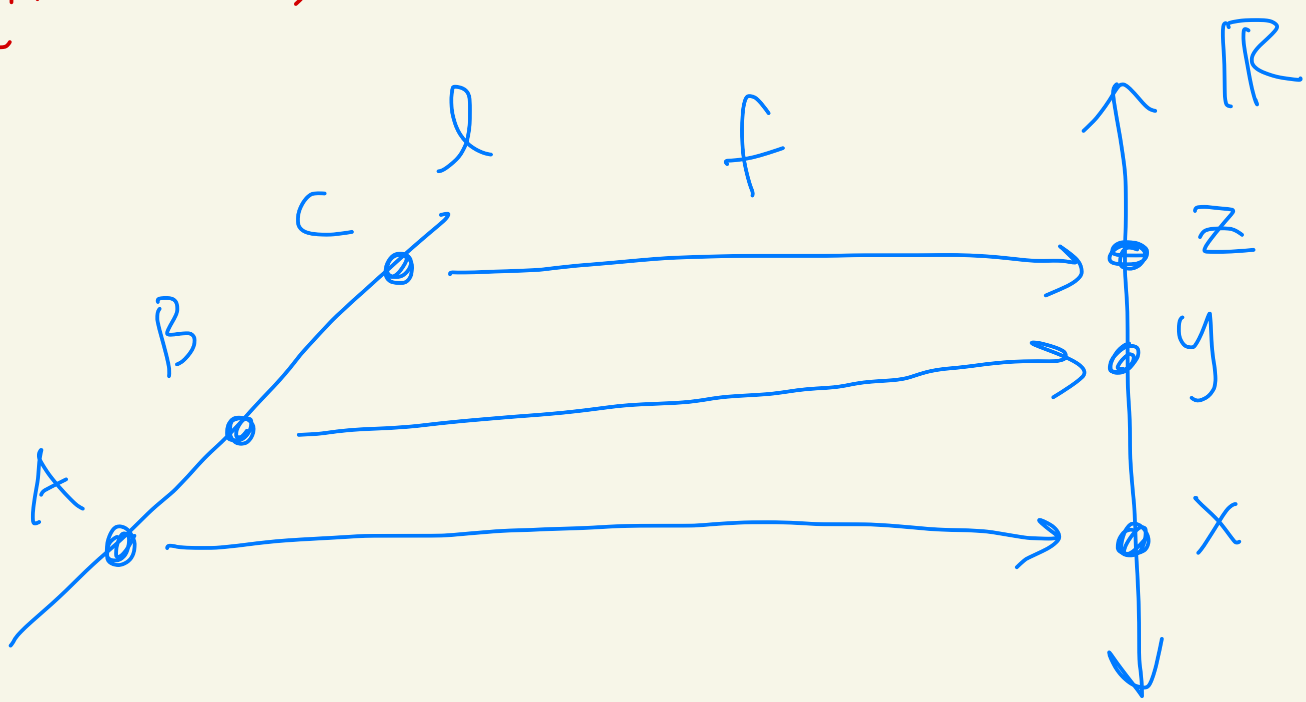
$-1<0<5$

Theorem: Let $(\mathcal{P}, \mathcal{L}, d)$ be a metric geometry.

Let ℓ be a line and $f : \ell \to \mathbb{R}$ be a ruler.

Let $A, B, C \in \ell$ with coordinates $x, y, z \in \mathbb{R}$

[That is, $x = f(A)$, $y = f(B)$, $z = f(C)$]



Then,

$A - B - C$ iff $x * y * z$.

$\boxed{\text{proof:}}$

($\Rightarrow$) Suppose $A-B-C$.

Then A, B, C are distinct by def.

Since f is a ruler, f is a bijection so x, y, z are distinct.

Since $A-B-C$ we know that $\boxed{AC = AB + BC}$.

Since f is a ruler we know that

$$AC = |f(A) - f(C)| = |x - z|$$

$$AB = |f(A) - f(B)| = |x - y|$$

$$BC = |f(B) - f(C)| = |y - z|$$

Thus,

$$\boxed{|x-z| = |x-y| + |y-z|} \quad (*)$$

We want to show that this implies that either $x<y<z$ or $z<y<x$ and thus $x*y*z$.

Since x,y,z are distinct there are 6 cases to consider:

(i) $x<y<z$ (ii) $z<y<x$
(iii) $y<x<z$ (iv) $z<x<y$
(v) $x<z<y$ (vi) $y<z<x$

We want (i) or (ii) to be true, so we have to rule out the other four cases.

Let's see how (iii) would lead to a contradiction for example.

Suppose (iii) $y < x < z$.

Then, $|x-y| = x-y$

$|y-z| = z-y$

$|x-z| = z-x$.

Plugging back into (*) gives:

$$(z-x) = (x-y) + (z-y)$$

This becomes $2y = 2x$

or $x = y$.

Contradiction since x, y, z are distinct.

Similar arguments for (iv), (v), and (vi) lead to contradictions.

Hence, (i) or (ii) is true.

So, $x * y * z$.

($\Leftarrow$) Suppose $x * y * z$.

So, either $x<y<z$ or $z<y<x$.

So x, y, z are distinct and since f is a bijection, A, B, C are distinct.

<u>Case 1:</u> Suppose $x<y<z$.

Then,
$$|x-y| = y-x$$
$$|y-z| = z-y$$
$$|x-z| = z-x$$

We know

$$\underbrace{(z-x)}_{|x-z|} = \underbrace{(y-x)}_{|x-y|} + \underbrace{(z-y)}_{|y-z|}$$

So, $|x-z| = |x-y| + |y-z|$

Thus,

$$|f(A)-f(C)| = |f(A)-f(B)| + |f(B)-f(C)|$$

Since f is a ruler we get that

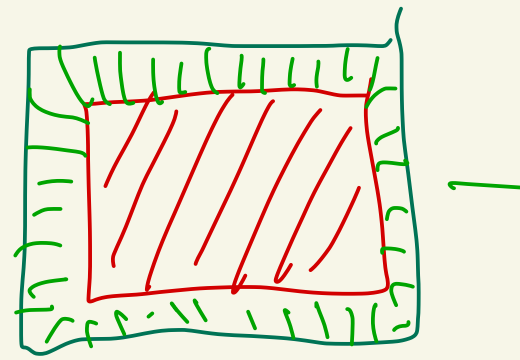
$$\underbrace{AC}_{d(A,C)} = \underbrace{AB}_{d(A,B)} + \underbrace{BC}_{d(B,C)}$$

Thus, $A-B-C$.

<u>Case 2:</u> Do a similar proof

for $z < y < x$ to show that $A-B-C$.

Thus, by case 1 and case 2 we have $A-B-C$.



<u>Corollary:</u> Let $(\mathcal{P}, \mathcal{L}, d)$ be a metric geometry. Let ℓ be a line and A, B, C be distinct points on ℓ. Then either $A-B-C$ or $A-C-B$ or $B-A-C$.

<u>proof:</u> HW 4 $\blacksquare$

<u>Theorem:</u> Consider the Euclidean plane $\mathcal{E} = (\mathbb{R}^2, \mathcal{L}_E, d_E)$. Let $A, B, C \in \mathbb{R}^2$. Then, $A-B-C$ if and only if there exists $t \in \mathbb{R}$ with

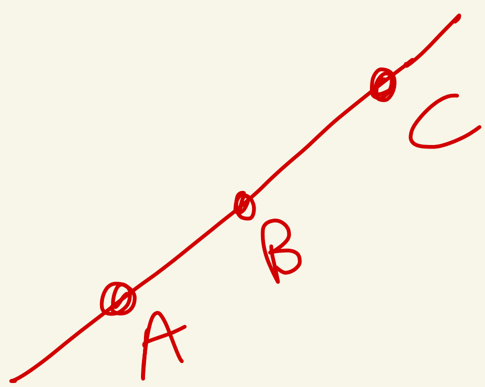
$0 < t < 1$ with $B = A + t(C - A)$.

<u>Proof</u>: HW 4 $\blacksquare$

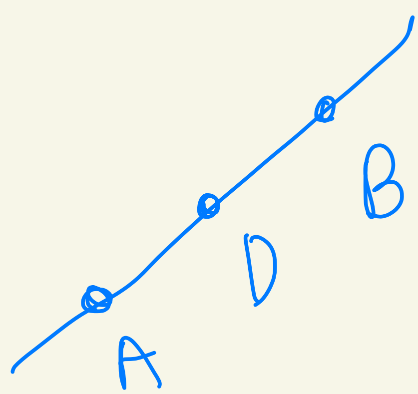
<u>Theorem</u>: Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry. Let A, B be distinct points in $\mathscr{P}$.

Then:

(i) there exists a point C where $A - B - C$



(ii) there exists a point D where $A - D - B$



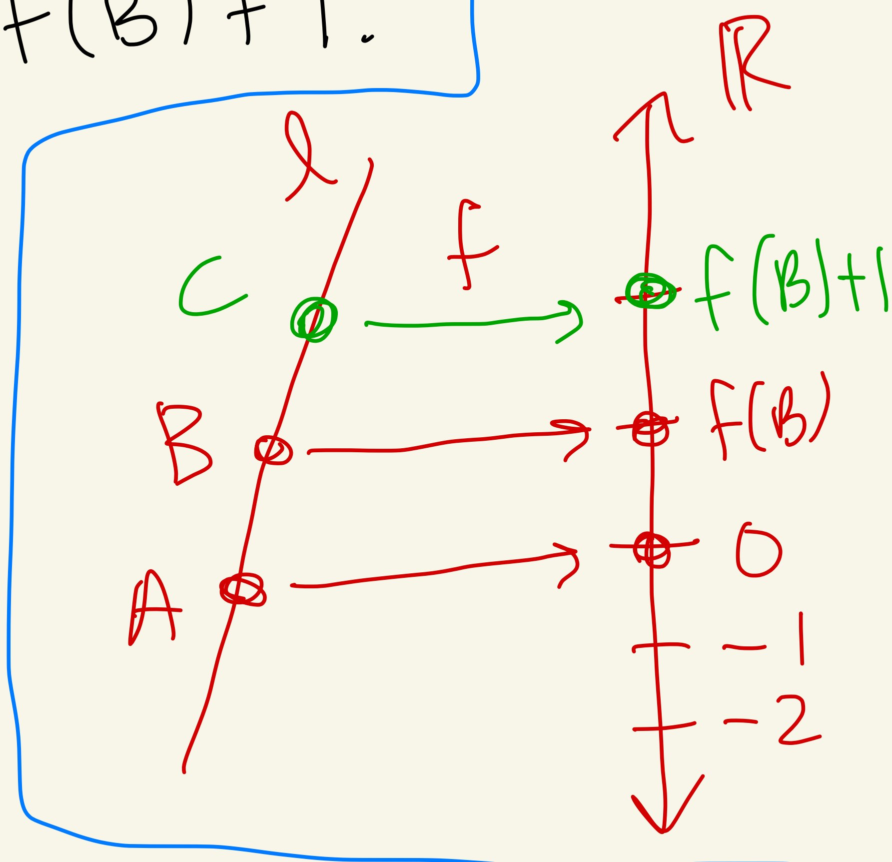
<u>proof:</u> Since $A \neq B$ there exists a unique line $\ell = \overleftrightarrow{AB}$.

Let $f : \ell \to \mathbb{R}$ be a ruler where $f(A) = 0$ and $f(B) > 0$.

this exists by a previous theorem

(i) Let $z = f(B) + 1$.

Since f is a bijection there exists a point $C \in \ell$ where $f(C) = f(B) + 1$

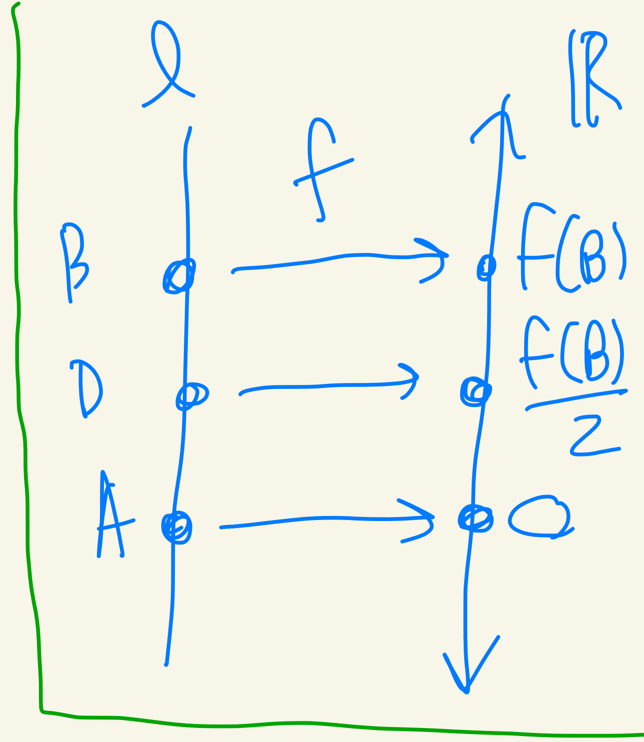


Then, $0 < f(B) < (f(B) + 1)$

So, $A - B - C$.

(ii) Let $w = \frac{f(B)}{2}$.

Since f is a bijection there exists $D \in \ell$ where $f(D) = \frac{f(B)}{2}$



So, $0 < \frac{f(B)}{2} < f(B)$

Thus, $f(A) < f(D) < f(B)$

Thus, $A - D - B$.