Math 4300
9/6/23

Topic 2 - Metric Geometries

Def: Let S be a set.

A **distance function** $d : S \times S \to \mathbb{R}$ is a function that satisfies the following for all $P, Q \in S$:

(i) $d(P,Q) \geq 0$

(ii) $d(P,Q) = 0$ iff $P = Q$

(iii) $d(P,Q) = d(Q,P)$

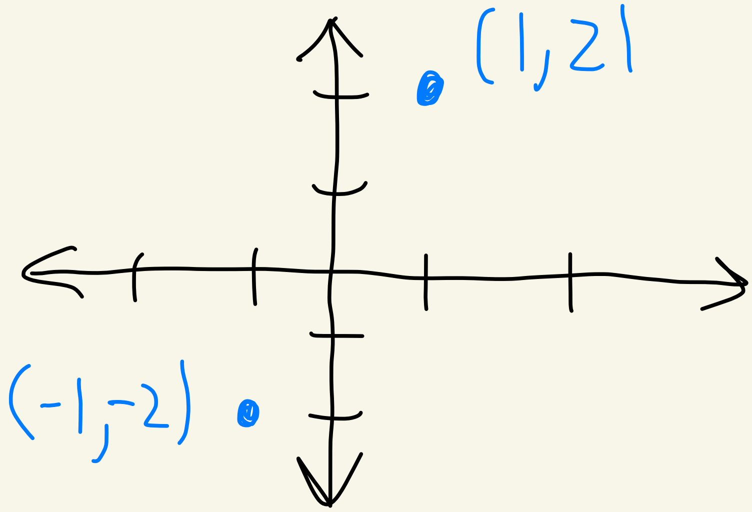
Ex: Define the **Euclidean distance** $d_E : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

as

$$d_E\big(\underbrace{(x_1,y_1)}_{P},\underbrace{(x_2,y_2)}_{Q}\big) =$$

$$= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

<u>Ex:</u>



$$d_E((1,2),(-1,-2))$$

$$= \sqrt{(1-(-1))^2+(2-(-2))^2}$$

$$= \sqrt{20} \approx 4.4721$$

<u>Theorem:</u> d_E is a distance function.

<u>proof:</u> Let $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ be in $\mathbb{R}^2$.

(i) $d(P,Q) = \sqrt{\underbrace{(x_1 - x_2)^2}_{\geqslant 0} + \underbrace{(y_1 - y_2)^2}_{\geqslant 0}} \geqslant 0$

(ii) $d(P,Q) = 0$

iff $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 0$

iff $\underbrace{(x_1 - x_2)^2}_{\geqslant 0} + \underbrace{(y_1 - y_2)^2}_{\geqslant 0} = 0$

iff $(x_1 - x_2)^2 = 0$ and $(y_1 - y_2)^2 = 0$

iff $x_1 - x_2 = 0$ and $y_1 - y_2 = 0$

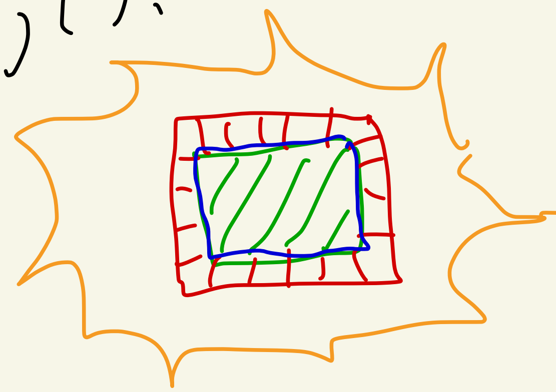
iff $x_1 = x_2$ and $y_1 = y_2$

iff $P = (x_1, y_1) = (x_2, y_2) = Q$

(iii) $d(P,Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$= d(Q,P).$$

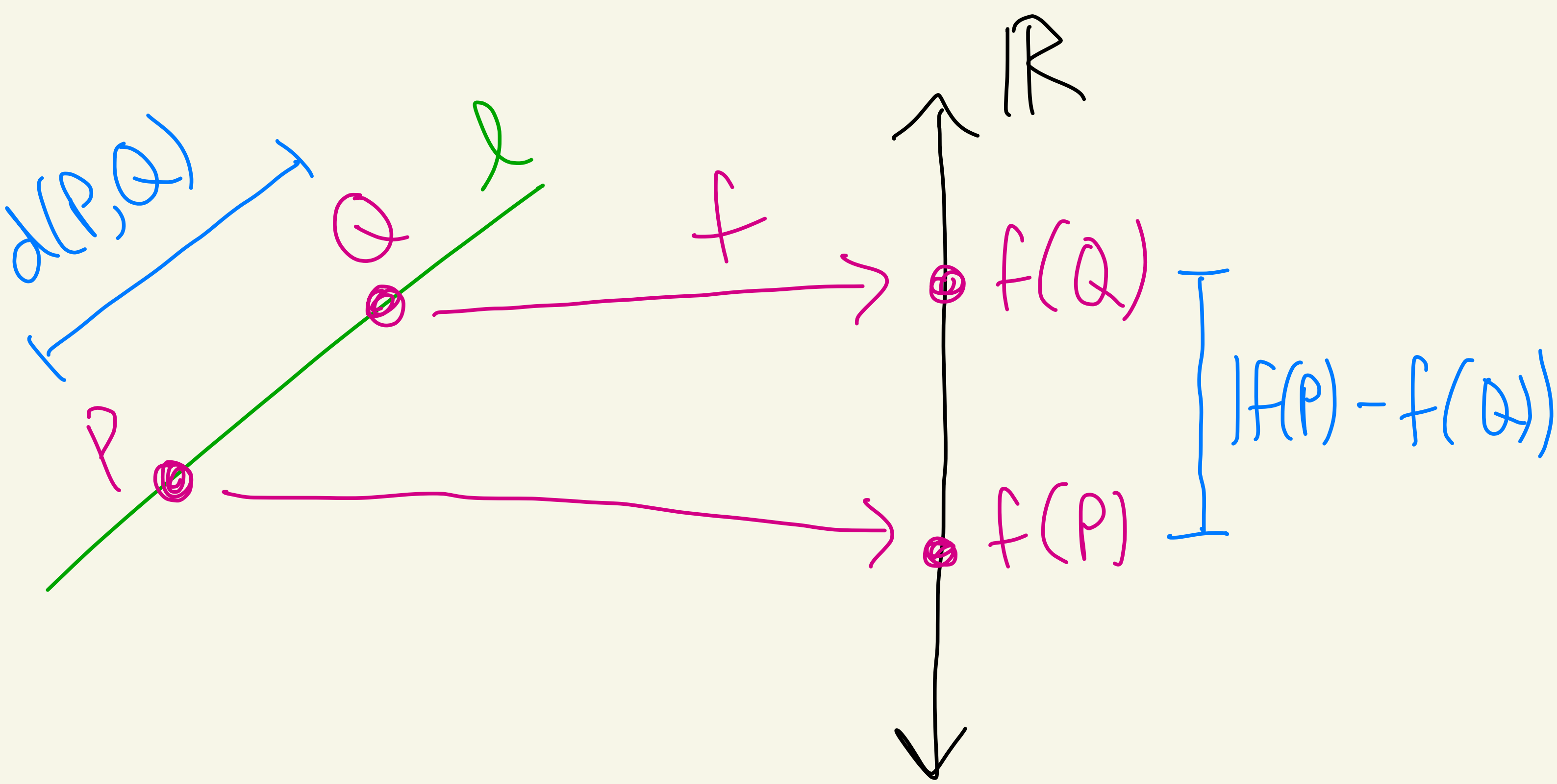


Def: Let $(\mathscr{P},\mathscr{L})$ be an incidence geometry. Let d be a distance function on $\mathscr{P}$. Let ℓ be a line from $\mathscr{L}$. A function $f:\ell\to\mathbb{R}$ is called a *ruler for ℓ* (or a *coordinate system* for ℓ) if the following are true:

(i) f is a bijection ($\underbrace{\text{onto}}_{\text{surjective}}$ & $\underbrace{1\text{-}1}_{\text{injective}}$) between ℓ and $\mathbb{R}$

(ii) $\overbrace{|f(P)-f(Q)|}^{\text{absolute value in } \mathbb{R}} = d(P,Q)$

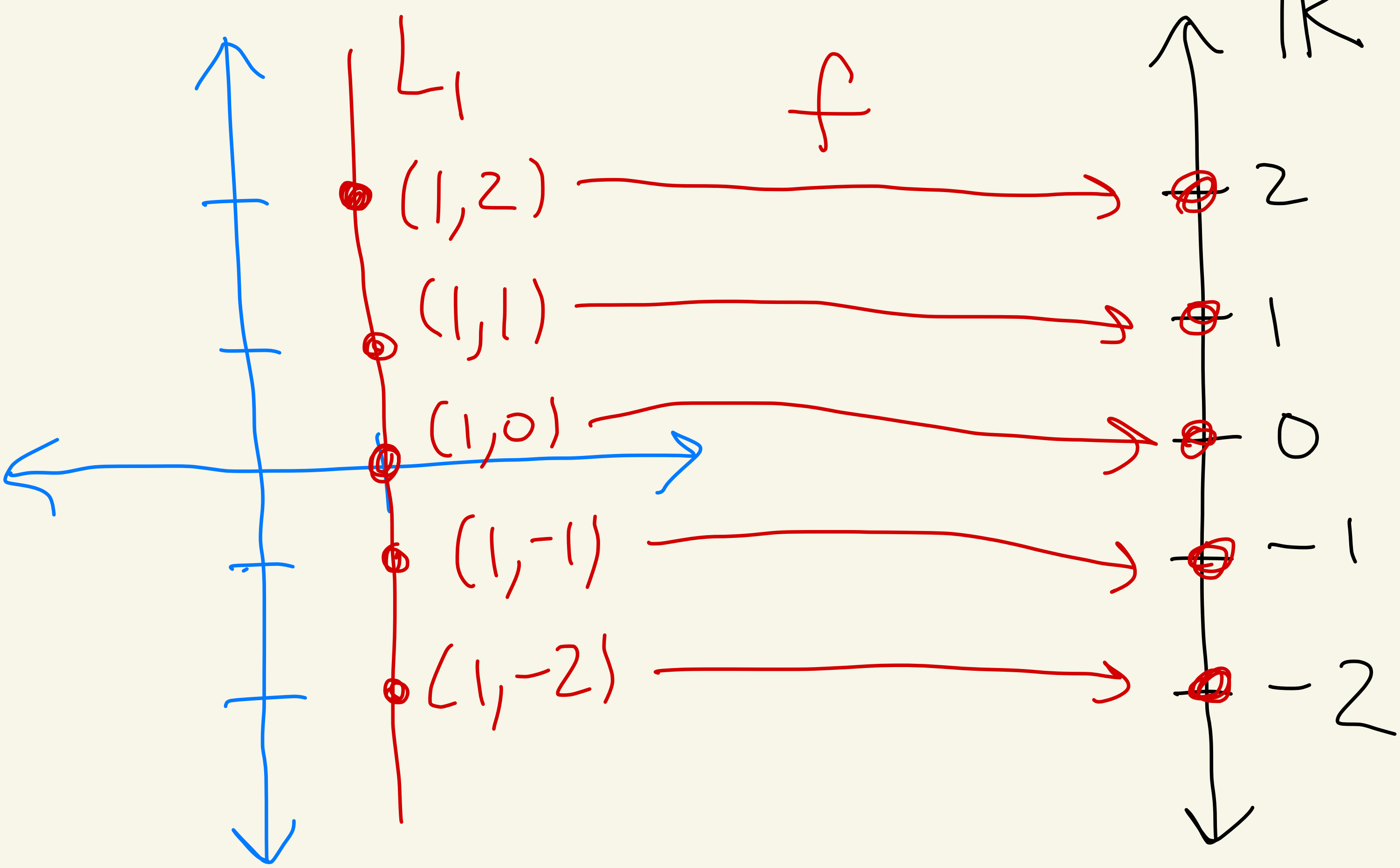
for all $P, Q \in \mathcal{P}$.



Ex: Consider the Euclidean plane $\mathcal{E} = (\underbrace{\mathbb{R}^2}_{\mathscr{P}}, \underbrace{\mathscr{L}_E}_{\mathscr{L}})$ with distance function d_E from above.

Let's look at L_1.

Define $f: L_1 \to \mathbb{R}$ by $f(1,y) = y$



Let's check that f is a ruler.

f is a bijection. We will prove soon in general.

If $P = (1, y_1)$ and $Q = (1, y_2)$ are on L_1.

Then,

$$d_E(P, Q) = \sqrt{\underbrace{(1-1)}_{0}{}^2 + (y_1 - y_2)^2}$$

$$= \sqrt{(y_1 - y_2)^2}$$

$$\boxed{\begin{array}{c} |z| = \sqrt{z^2} \\ z \in \mathbb{R} \end{array}}$$

$$= |y_1 - y_2|$$

$$= |f(1, y_1) - f(1, y_2)|$$

$$= |f(P) - f(Q)|$$

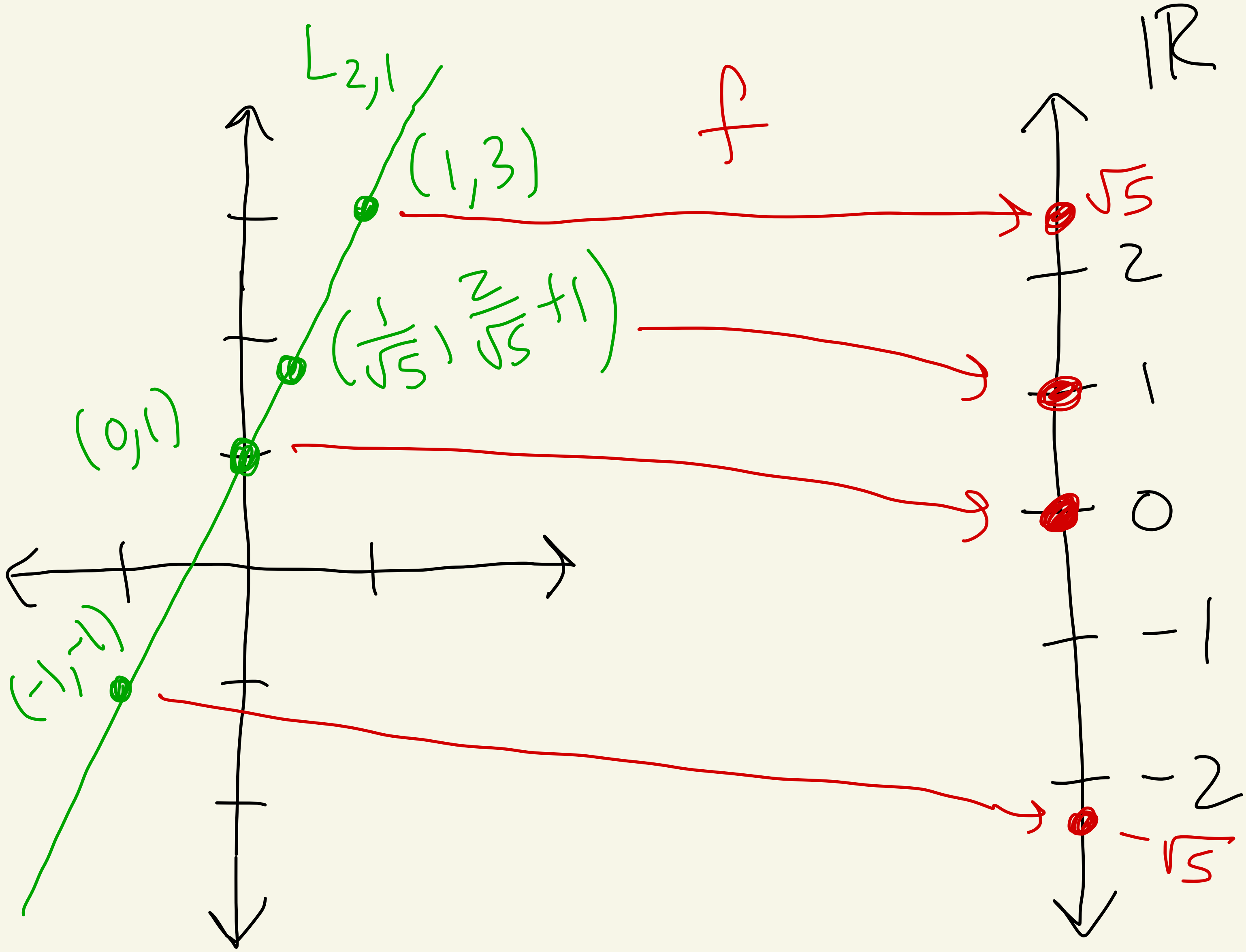
$\blacksquare$

Now consider the line

$L_{2,1} = \{(x,y) \in \mathbb{R}^2 \mid y = 2x+1\}$

Define $f : L_{2,1} \to \mathbb{R}$

by $f(x,y) = \sqrt{5} \cdot x$



One can show this is a ruler. We will prove soon. For example,

$$d_E((-1,-1),(0,1)) = \sqrt{(-1-0)^2+(-1-1)^2}$$
$$= \sqrt{5}$$
$$= |-\sqrt{5}-0|$$
$$= |f(-1,-1)-f(0,1)|$$

Lemma: Let $(\mathscr{P}, \mathscr{L})$ be an incidence geometry.

Let d be a distance function on $\mathscr{P}$.

Let ℓ be a line in $\mathscr{L}$.

Suppose $f : \ell \to \mathbb{R}$ and

f is onto and

$|f(P) - f(Q)| = d(P,Q)$ for all $P, Q \in \mathscr{P}$.

Then, f is 1-1 and hence is a ruler on ℓ.

<u>proof:</u> Let's show the above conditions imply that f is 1-1.

Let $P, Q \in \mathscr{P}$ and $f(P) = f(Q)$.

We must show that $P = Q$.

We have that

$$d(P,Q) \underset{\text{assumption}}{=} |f(P) - f(Q)| \underset{\text{since } f(P)=f(Q)}{=} 0$$

Since $d(P,Q) = 0$ and d is a distance function we know $P = Q$.

Thus, f is 1-1.

So, f is a ruler for ℓ.