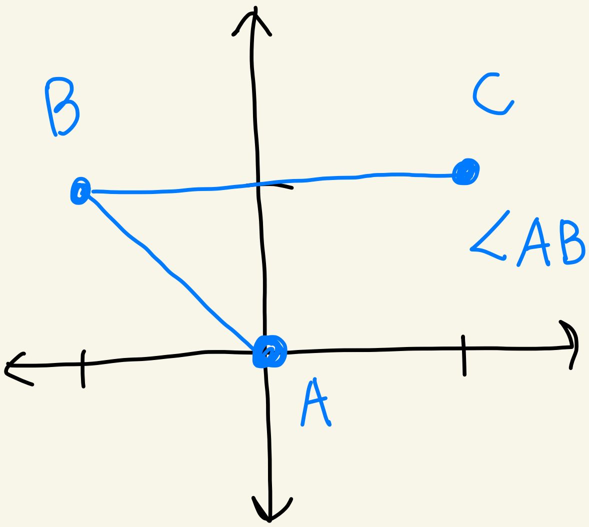
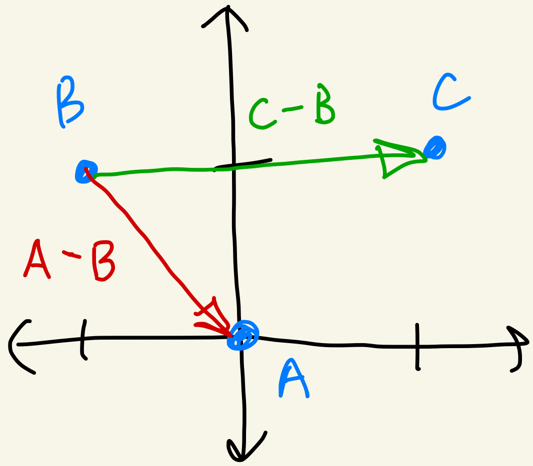
Math 4300
Homework 10
Solutions

① $A=(0,0)$, $B=(-1,1)$, $C=(1,1)$.





$$m_E(\angle ABC) = \cos^{-1}\left(\frac{\langle C-B, A-B\rangle}{\|C-B\|\cdot\|A-B\|}\right)$$

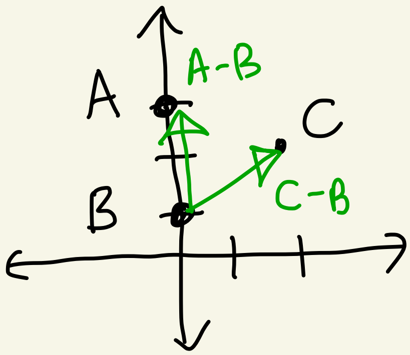
$$= \cos^{-1}\left(\frac{\langle (2,0), (1,-1)\rangle}{\|(2,0)\|\cdot\|(1,-1)\|}\right)$$

$$= \cos^{-1}\left(\frac{2+0}{\sqrt{2^2+0^2}\,\sqrt{1^2+(-1)^2}}\right)$$

$$= \cos^{-1}\left(\frac{2}{2\cdot\sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^{\circ}$$

② (a) $\boxed{A=(0,3),\ B=(0,1),\ C=(\sqrt{3},2)}$

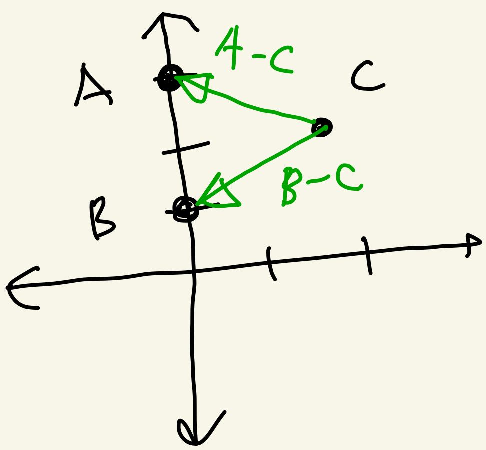


$$m_E(\angle ABC) = \cos^{-1}\left(\frac{\langle A-B, C-B\rangle}{\|A-B\|\cdot\|C-B\|}\right)$$

$$= \cos^{-1}\left(\frac{\langle(0,2),(\sqrt{3},1)\rangle}{\|(0,2)\|\cdot\|(\sqrt{3},1)\|}\right)$$

$$= \cos^{-1}\left(\frac{0+2}{\sqrt{0^2+2^2}\ \sqrt{(\sqrt{3})^2+1^2}}\right)$$

$$= \cos^{-1}\left(\frac{2}{2\cdot 2}\right) = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

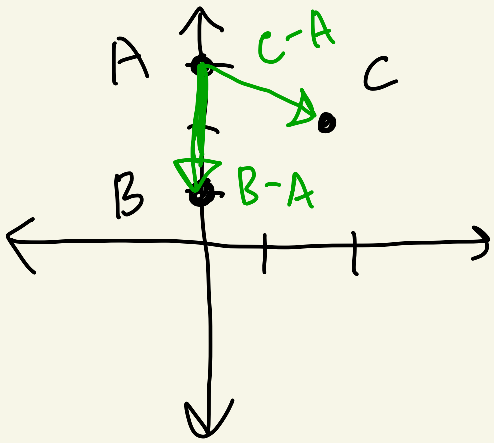


$$m_E(\angle BCA) = \cos^{-1}\left(\frac{\langle B-C, A-C\rangle}{\|B-C\|\cdot\|A-C\|}\right)$$

$$= \cos^{-1}\left(\frac{\langle(-\sqrt{3},-1),(-\sqrt{3},1)\rangle}{\|(-\sqrt{3},-1)\|\cdot\|(-\sqrt{3},1)\|}\right)$$

$$= \cos^{-1}\left(\frac{3+(-1)}{\sqrt{(-\sqrt{3})^2+(-1)^2}\cdot\sqrt{(-\sqrt{3})^2+(1)^2}}\right)$$

$$= \cos^{-1}\left(\frac{2}{2\cdot 2}\right) = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$



$$m_E(\angle CAB) = \cos^{-1}\left(\frac{\langle C-A, B-A\rangle}{\|C-A\|\cdot\|B-A\|}\right)$$

$$= \cos^{-1}\left(\frac{\langle(\sqrt{3},-1),(0,-2)\rangle}{\|(\sqrt{3},-1)\|\cdot\|(0,-2)\|}\right)$$

$$= \cos^{-1}\left(\frac{0 + 2}{\sqrt{(\sqrt{3})^2+(-1)^2}\cdot\sqrt{0^2+(-2)^2}}\right)$$

$$= \cos^{-1}\left(\frac{2}{2\cdot 2}\right) = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

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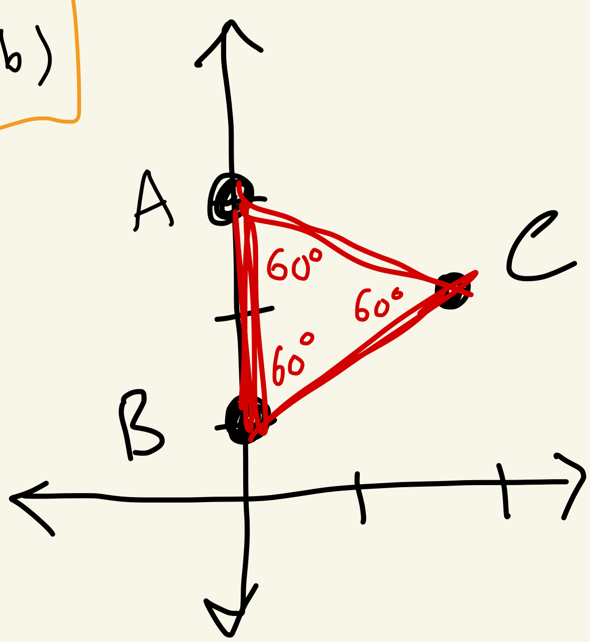
2 (b)



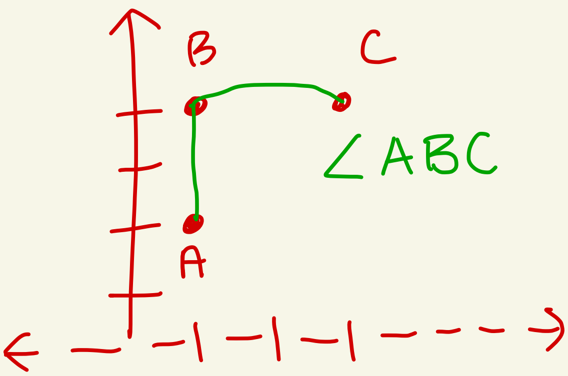
$$m_E(\angle ABC) + m_E(\angle BCA) + m_E(\angle CAB)$$

$$= 60^\circ + 60^\circ + 60^\circ = 180^\circ$$

③(a) $A=(1,2)$, $B=(1,4)$, $C=(3,4)$

From HW 6 we have that

$\overleftrightarrow{AB} = {}_1L$ and $\overleftrightarrow{BC} = {}_2L_{\sqrt{17}}$



We need T_{BA} and T_{BC}.

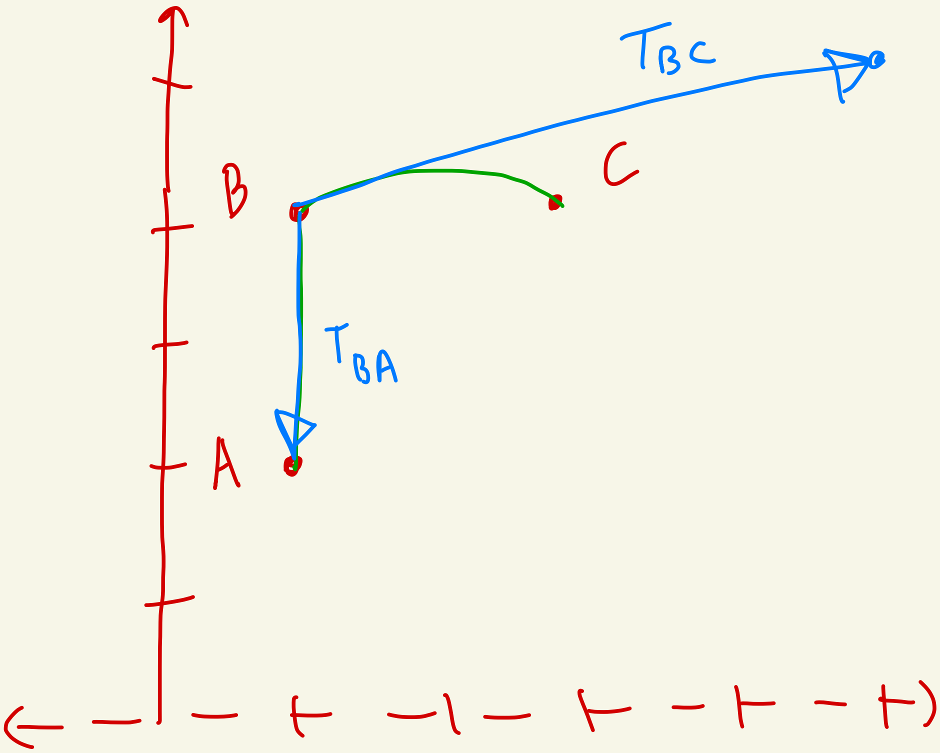
Since $\overleftrightarrow{BA} = {}_1L$ is a vertical line we have

$$T_{BA} = (0, y_A - y_B) = (0, 2-4) = (0, -2)$$

Since $\overleftrightarrow{BC} = {}_2L_{\sqrt{17}}$ is a non-vertical line and the vertex B is to the left of C, ie $x_B < x_C$ we have

$$T_{BC} = (y_B, c - x_B) = (4, 2-1) = (4, 1)$$

The picture is below.

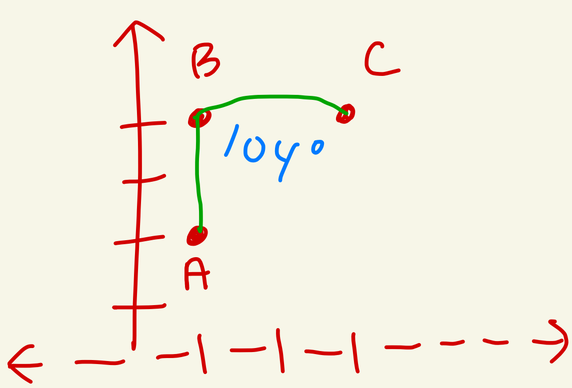


$$m_H(\angle ABC) = \cos^{-1}\left(\frac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\|\cdot\|T_{BC}\|}\right)$$

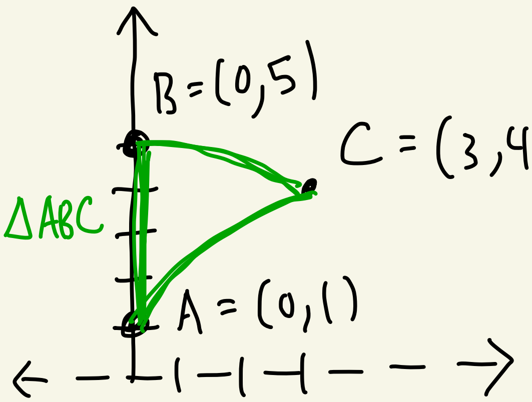
$$= \cos^{-1}\left(\frac{\langle(0,-2),(4,1)\rangle}{\|(0,-2)\|\cdot\|(4,1)\|}\right) = \cos^{-1}\left(\frac{0-2}{\sqrt{0^2+(-2)^2}\sqrt{4^2+1^2}}\right)$$

$$= \cos^{-1}\left(\frac{-2}{2\cdot\sqrt{17}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{17}}\right)$$

$$\approx \cos^{-1}(-0.2425356\ldots) \approx 104^\circ$$



④ (a)



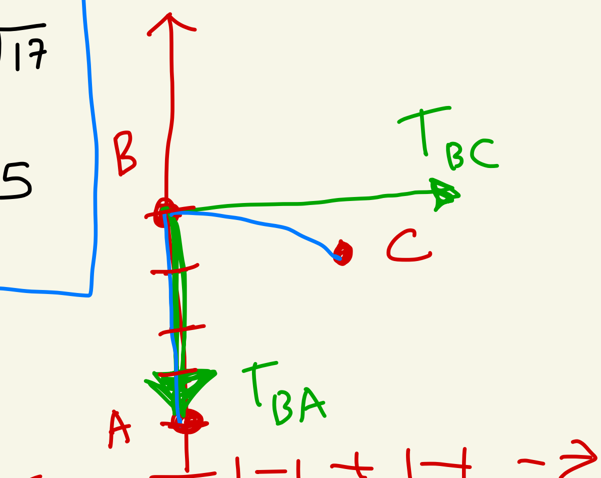
First we need the lines that these points lie on.

One can show that

$\overleftrightarrow{AB} = {}_0L$

$\overleftrightarrow{AC} = {}_4L_{\sqrt{17}}$

$\overleftrightarrow{BC} = {}_0L_5$



Let's calculate $m_H(\angle ABC)$.

We need T_{BA} and T_{BC}.

$\overleftrightarrow{BA}$ is a vertical line so

$$T_{BA} = (0, y_A - y_B) = (0, 1-5) = (0,-4)$$

$\overleftrightarrow{BC} = {}_0L_5$ is not a vertical line and $x_B < x_C$ so

$$T_{BC} = (y_B, c - x_B) = (5, 0-0) = (5,0).$$

B the vertex is to the left of C

Then,

$$m_H(\angle ABC) = \cos^{-1}\left(\frac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\| \cdot \|T_{BC}\|}\right)$$

$$= \cos^{-1}\left(\frac{\langle (0,-4), (5,0)\rangle}{\|(0,-4)\| \cdot \|(5,0)\|}\right) =$$

$$= \cos^{-1}\left(\frac{0+0}{4 \cdot 5}\right) = \cos^{-1}(0) = 90^\circ$$

$\boxed{\text{Let's calculate } m_H(\angle BCA)}$

C the center is to the right of B

We need T_{CB} and T_{CA} since C is the vertex.

since C is the vertex of $\angle BCA$.

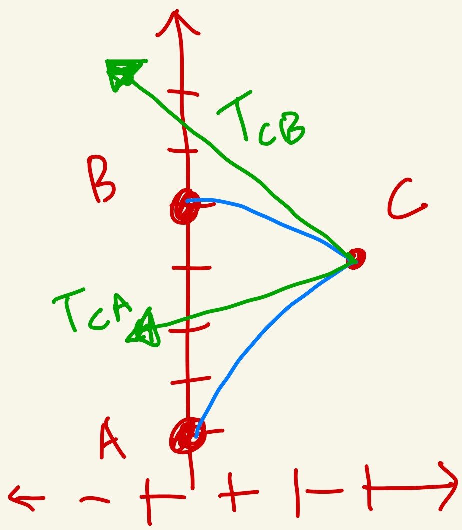
$\overleftrightarrow{CB} = {}_0L_5$ is a non-vertical line and $x_B < x_C$

so, $T_{CB} = -(y_c, c - x_c) = -(4, 0-3) = (-4, 3)$

$\overleftrightarrow{CA} = {}_4L_{\sqrt{17}}$ is a non-vertical line and $x_A < x_C$

so $T_{CA} = -(y_c, c - x_c) = -(4, 4-3)$
$= (-4, -1)$

C the vertex is to the right of A



Thus,

$$m_H(\angle BCA) = \cos^{-1}\left(\frac{\langle T_{CB}, T_{CA}\rangle}{\|T_{CB}\|\cdot\|T_{CA}\|}\right)$$

$$= \cos^{-1}\left(\frac{\langle(-4,3),(-4,-1)\rangle}{\|(-4,3)\|\cdot\|(-4,-1)\|}\right)$$

$$= \cos^{-1}\left(\frac{16-3}{\sqrt{16+9}\cdot\sqrt{16+1}}\right)$$

$$= \cos^{-1}\left(\frac{13}{5\cdot\sqrt{17}}\right)$$

$$\approx \cos^{-1}(0,630592625\ldots)$$

$$\approx 50.91^{\circ}$$

$\boxed{\text{Let's calculate } m_H(\angle CAB)}$

A is the vertex so we need T_{AC} and T_{AB}.

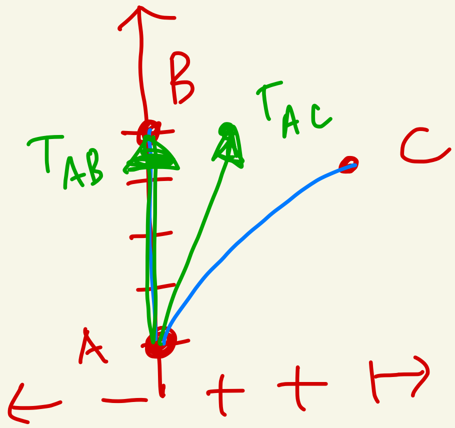
the vertex A is to the left of C

$\overleftrightarrow{AC} = {}_4L_{\sqrt{17}}$ and $x_A < x_C$

so $T_{AC} = (y_A, c - x_A) = (1, 4-0) = (1,4)$

$\overleftrightarrow{AB} = {}_0L$ is a vertical line

so $T_{AB} = (0, y_B - y_A) = (0, 5-1) = (0,4)$



Thus,

$$m_H(\angle CAB) = \cos^{-1}\left(\frac{\langle T_{AC}, T_{AB}\rangle}{\|T_{AC}\|\cdot\|T_{AB}\|}\right)$$

$$= \cos^{-1}\left(\frac{\langle (1,4),(0,4)\rangle}{\|(1,4)\|\cdot\|(0,4)\|}\right)$$

$$= \cos^{-1}\left(\frac{0+16}{\sqrt{1+16}\ \sqrt{0+16}}\right)$$

$$= \cos^{-1}\left(\frac{16}{\sqrt{17}\cdot 4}\right)$$

$$\approx \cos^{-1}(0.970142 5\ldots)$$

$$\approx 14.04^\circ$$

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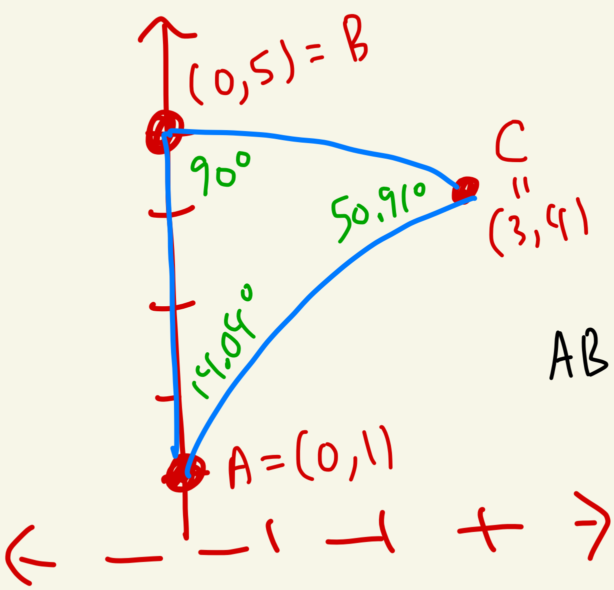
4 (b)



Here we have a 90° at $\angle ABC$.

Let's find the side lengths of the triangle.

$$AB = d_H(A,B) = \left|\ln\left(\frac{1}{4}\right)\right| \approx 1.38629$$

$$BC = d_H(B,C) = \left|\ln\left(\frac{(0-0+5)/5}{(3-0+5)/4}\right)\right| = \left|\ln\left(\frac{1}{2}\right)\right|$$

$$\boxed{\overleftrightarrow{BC} = {}_0L_5}$$

$$\approx 0.693147\ldots$$

$$AC = d_H(A,C) = \left|\ln\left(\frac{(0-4+\sqrt{17})/1}{(3-4+\sqrt{17})/4}\right)\right| = \left|\ln\left(\frac{-4+\sqrt{17}}{-\frac{1}{4}+\frac{\sqrt{17}}{4}}\right)\right|$$

$$\boxed{\overleftrightarrow{AC} = {}_4L_{\sqrt{17}}}$$

$$\approx 1.847246\ldots$$

Thus,

$$AB^2 + BC^2 \approx 2.402265\ldots$$

and

$$AC^2 \approx 3.4123\ldots$$

So, $AB^2 + BC^2 \neq AC^2$.

$\boxed{4(c)}$ $m_H(\angle ABC) + m_H(\angle BCA) + m_H(\angle CAB)$

$$\approx 90^\circ + 50.91^\circ + 14.04^\circ$$

$$\approx 154.95^\circ \leftarrow \boxed{\text{less than } 180^\circ}$$