Math 4460

2/20/23

Proof of theorem from Weds:

Let $a, b, c \in \mathbb{Z}$.

Suppose a and b are not both zero.

Let $d = \gcd(a,b)$.

1 ($\Rightarrow$) Suppose there exist $x, y \in \mathbb{Z}$ where $ax + by = c$

The goal will be to show that $d \mid c$.

Since $d = \gcd(a,b)$ we know $d \mid a$ and $d \mid b$.

Thus, $a = dk$ and $b = dk'$ where $k, k' \in \mathbb{Z}$.

Hence,

$$\begin{aligned} c &= ax + by \\ &= dkx + dk'y \\ &= d\underbrace{[kx + k'y]}_{\text{this is an integer because } k,x,k',y \in \mathbb{Z}} \end{aligned}$$

$\therefore \quad d \mid c$

($\Leftarrow$) Now assume $d \mid c$.

Our goal is to find $x, y \in \mathbb{Z}$ where $ax + by = c$.

Since $d \mid c$ we know $c = dq$ where $q \in \mathbb{Z}$

We know there exist $x_0, y_0 \in \mathbb{Z}$ where $d = ax_0 + by_0$ (thm in class)

Multiply by q to get

$$\underbrace{dq}_{c} = a(qx_0) + b(qy_0)$$

Thus,

$$c = ax + by$$

where $x = qx_0$ and $y = qy_0$. $\boxed{1}$

(2) Suppose $ax+by=c$ has integer solutions and $x_0, y_0 \in \mathbb{Z}$ is a particular solution.

That is, $ax_0 + by_0 = c$.

Our goal is to show that all the solutions are of the form

$$\left.\begin{aligned} x &= x_0 - t\left(\frac{b}{d}\right) \\ y &= y_0 + t\left(\frac{a}{d}\right) \end{aligned}\right\} \quad (*)$$

where $t \in \mathbb{Z}$.

First let's check that $(*)$ gives a solution to $ax+by=c$ by plugging it in to the left side.

We have that

$$\underbrace{a\left(x_0 - t\frac{b}{d}\right)}_{x} + b\underbrace{\left(y_0 + t\frac{a}{d}\right)}_{y}$$

$$= ax_0 - \cancel{t\frac{ab}{d}} + by_0 + \cancel{t\frac{ab}{d}}$$

$$= ax_0 + by_0$$

particular solution

$$= c$$

But why does $(*)$ give us all the solutions?

We are assuming $\boxed{ax_0 + by_0 = c.}$

Suppose $x, y \in \mathbb{Z}$ is another solution, that is, $\boxed{ax + by = c.}$

Subtracting these two equations gives

$$a(x-x_0)+b(y-y_0)=0$$

Dividing by d gives

$$\frac{a}{d}(x-x_0)+\frac{b}{d}(y-y_0)=0$$

$\frac{a}{d}, \frac{b}{d} \in \mathbb{Z}$ because $d|a$ and $d|b$

Thus,

$$\frac{a}{d}(x-x_0)=\frac{-b}{d}(y-y_0)$$

Multiplying by -1 gives

$$\boxed{\frac{a}{d}(x_0 - x) = \frac{b}{d}(y - y_0)} \quad (**)$$

$(**)$ tells us that $\frac{a}{d} \mid \frac{b}{d}(y-y_0)$

We know from a previous theorem that $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

Thus, from another previous theorem, since $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$ and $\frac{a}{d} \mid \frac{b}{d}(y-y_0)$ we know that $\frac{a}{d} \mid (y-y_0)$.

Hence, thus, ergo there exists $t \in \mathbb{Z}$ where $y - y_0 = t\left(\frac{a}{d}\right)$.

So, $\boxed{y = y_0 + t\left(\frac{a}{d}\right)}$

Plug $y - y_0 = t\left(\frac{a}{d}\right)$ into $(**)$ to get that

$$\frac{a}{d}(x_0 - x) = \frac{b}{d}\Big(\underbrace{t\left(\tfrac{a}{d}\right)}_{y-y_0}\Big)$$

Thus,

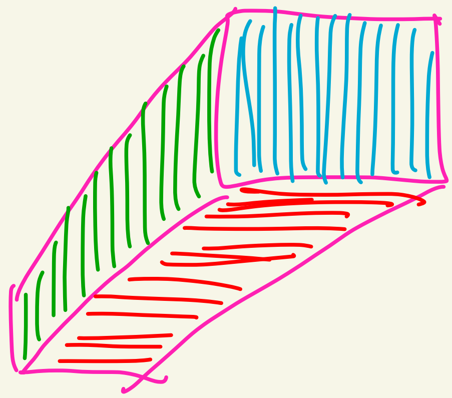
$$x_0 - x = t\left(\frac{b}{d}\right)$$

Hence,

$$\boxed{x = x_0 - t\left(\frac{b}{d}\right)}$$

Thus, every solution to $ax+by=c$ is of the form $(*)$. $\boxed{2}$

3 Follows from 1 and 2



Ex: (HW 2 #4(f))

Solve $\boxed{39x + 17y = 22}$

What is $\gcd(39, 17)$?

$$\boxed{\begin{aligned} 39 &= 2 \cdot 17 + 5 \\ 17 &= 3 \cdot 5 + 2 \\ 5 &= 2 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned}}$$

$\boxed{\gcd(39, 17) = 1}$

Since $\gcd(39, 17) = 1$ divides 22 there must exist integer solutions to $39x + 17y = 22$

Let's find a particular solution.
Solve the first 3 equations
above for the remainders.

$$5 = 1 \cdot 39 - 2 \cdot 17$$
$$2 = 1 \cdot 17 - 3 \cdot 5$$
$$1 = 1 \cdot 5 - 2 \cdot 2$$

Start here

So,

gcd(39, 17)

$$1 = 1 \cdot \boxed{5} - 2 \cdot \boxed{2}$$
$$= 1 \cdot (1 \cdot \boxed{39} - 2 \cdot \boxed{17}) - 2 \cdot (1 \cdot \boxed{17} - 3 \cdot \boxed{5})$$
$$= 1 \cdot \boxed{39} - 4 \cdot \boxed{17} + 6 \cdot \boxed{5}$$

$$= 1\cdot\boxed{39} - 4\cdot\boxed{17} + 6\cdot(1\cdot\boxed{39} - 2\cdot\boxed{17})$$

$$= 7\cdot\boxed{39} - 16\cdot\boxed{17}$$

Thus,

$$\boxed{39(7) + 17(-16) = 1}$$

Multiply by 22 to get

$$\boxed{39\underbrace{(154)}_{7\cdot 22} + 17\underbrace{(-352)}_{-16\cdot 22} = 22}$$

Thus a particular solution to

$$39x + 17y = 22$$

is $\boxed{x_0 = 154, \ y_0 = -352}$

Thus, every solution to

$39x + 17y = 22$ ← ax+by=c
d=1
c=22
a=39
b=17

is of the form

$$x = \underbrace{154 - t\left(\frac{17}{1}\right)}_{x_0 - t\left(\frac{b}{d}\right)} = 154 - 17t$$

$$y = \underbrace{-352 + t\left(\frac{39}{1}\right)}_{y_0 + t\left(\frac{a}{d}\right)} = -352 + 39t$$

Plug in different t such as $t = 0, 1, -1, 2, -2, \ldots$ to get some solutions. $\blacksquare$