4460
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This theorem will be used for Pythagorean triples.

Theorem:

Let $a, b \in \mathbb{Z}$ with $a \geq 1$, $b \geq 1$.
Suppose $\gcd(a,b) = 1$ and

$$ab = c^n$$

where $c, n \in \mathbb{Z}$, $c \geq 1$, $n \geq 1$.
Then, there exists $d, e \in \mathbb{Z}$ with

$$a = d^n \quad \text{and} \quad b = e^n$$

and $d \geq 1$, $e \geq 1$

Ex: $\underbrace{9}_{a} \cdot \underbrace{16}_{b} = \underbrace{12^2}_{c^n}$

$\gcd(a,b) = 1$

$a = 3^2 = d^2$

$b = 4^2 = e^2$

proof: Suppose $\gcd(a,b)=1$ and $ab=c^n$ where $a,b,c,n \geq 1$.

If $a=1$, then $b=c^n$
Here set $d=1, e=c$.

If $b=1$, then $a=c^n$.
Set $d=c, e=1$.

So assume $a,b \geq 2$.
That also makes $c \geq 2$. ← because $ab=c^n$

Since $\gcd(a,b)=1$ we know that the prime factors of a and b are distinct,

Ex:
$$a = \underbrace{13^4 \cdot 2^6}_{p_1^{a_1} \, p_2^{a_2}}$$
$$b = 5^2 \cdot 7^8 \cdot 11^{10}$$

Thus we may write
$$a = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$$

$$b = p_{r+1}^{a_{r+1}} p_{r+2}^{a_{r+2}} \cdots p_{r+s}^{a_{r+s}}$$

$p_3^{a_3} p_4^{a_4} p_5^{a_5}$

Where $p_1, p_2, \ldots, p_r, p_{r+1}, \ldots, p_{r+s}$ are distinct primes and $a_1, \ldots, a_{r+s} \geq 1$, $r \geq 1$, $s \geq 1$

Ex!

$ab = (2^3 \cdot 5^1 \cdot 7^4 \cdot 11^5 \cdot 13^3)^2$
$= c^2$

Factor c:

$$c = q_1^{b_1} q_2^{b_2} \cdots q_k^{b_k}$$

where $q_1, q_2, \ldots, q_k$ are primes and $b_i \geq 1$.

Plug everything into $ab = c^n$:

$$\underbrace{p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}}_{a} \underbrace{p_{r+1}^{a_{r+1}} \cdots p_{r+s}^{a_{r+s}}}_{b} = \underbrace{q_1^{nb_1} q_2^{nb_2} \cdots q_k^{nb_k}}_{c^n}$$

By FTOA, the factorizations on both sides in the equation above have to be the same.
That is, $r+s=k$, and the q_k are the same as the p_i (except up to reordering possibly), and the corresponding exponents are the same.
Thus, we may reorder/renumber the q's so that

$q_j = p_j$ for all j

And then $a_j = nb_j$ for all j.
Then,

$$a = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} = q_1^{nb_1} q_2^{nb_2} \cdots q^{nb_r}$$
$$= (q_1^{b_1} q_2^{b_2} \cdots q_r^{b_r})^n$$

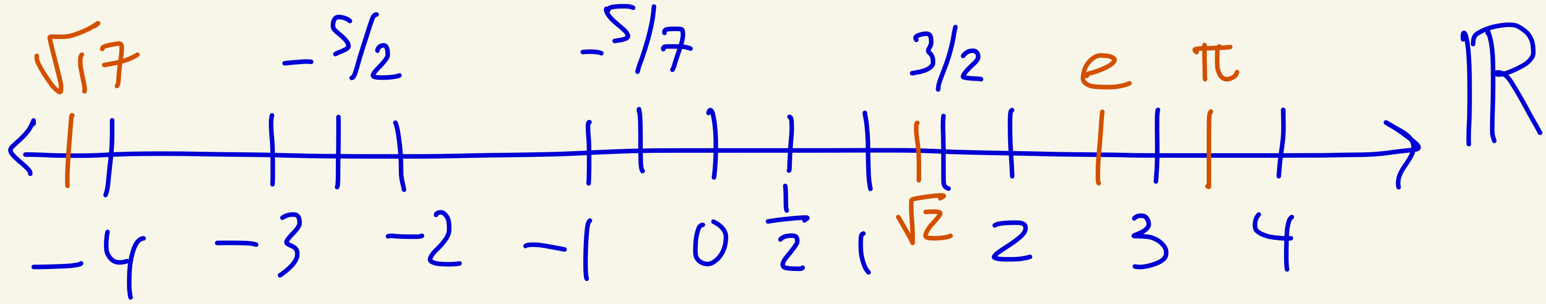
d^n

and

$$b = p_{r+1}^{a_{r+1}} \cdots p_{r+s}^{a_{r+s}} = q^{nb_{r+1}} \cdots q^{nb_{r+s}}$$
$$= \underbrace{\left(q^{b_{r+1}} \cdots q^{b_{r+s}}\right)^n}_{e^n}$$

$\boxed{\text{Rationals and irrationals recap}}$

The real number line $\mathbb{R}$ consists of all numbers with decimal expansions.



The real number line consists of the rational numbers $\mathbb{Q}$ and the irrational numbers $\mathbb{R}-\mathbb{Q}$

$\mathbb{Q}$ consists of all $\frac{x}{y}$ where $x, y \in \mathbb{Z}$ and $y \neq 0$. fractions

irrational #s are $\mathbb{R}-\mathbb{Q}$ that is any real number that's not a fraction/rational #

Ex:

rational	irrational
$1/2$, $\frac{-10}{13}$	$\sqrt{2}$
$1 = 1/1$	e
$-5 = -5/1$	π

$\underbrace{\qquad\qquad}_{\text{countable}}$ $\underbrace{\qquad\qquad}_{\text{uncountable}}$

HW 3
1(a)

Given $a, b \in \mathbb{Z}$ with $b \neq 0$, there exist $x, y \in \mathbb{Z}$ with $y \neq 0$ and

$$\gcd(x, y) = 1 \text{ and } \frac{a}{b} = \frac{x}{y}$$

<u>Ex:</u> $a = 25$, $b = 10$

$$\frac{a}{b} = \frac{25}{10} = \frac{5}{2} = \frac{x}{y}$$

$$x = 5,\ y = 2,\ \gcd(x, y) = 1$$

<u>Proof:</u> Let $a, b \in \mathbb{Z}$, $b \neq 0$.

Set $d = \gcd(a, b)$.

Set $x = \frac{a}{d}$, $y = \frac{b}{d}$.

Then by a theorem from class, $\gcd(x,y) = \gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

Furthermore,

$$\frac{a}{b} = \frac{a/d}{b/d} = \frac{x}{y}$$

$\blacksquare$

HW 3 1(d) Let p be a prime. Then, $\sqrt{p}$ is irrational.

<u>proof</u>: We will show that $\sqrt{p}$ is not a rational number.

Do this by contradiction.

Suppose $\sqrt{p}$ is a rational number.

Then, by the previous result,

$$\boxed{\sqrt{p} = \frac{x}{y}}$$ where

$x, y \in \mathbb{Z}$, $y \neq 0$ and $\gcd(x,y) = 1$.

Square the equation to get

$$p = \frac{x^2}{y^2}$$ which gives

$$\boxed{py^2 = x^2} \quad (*)$$

So, $p \mid x^2$.

Since p is prime and $p \mid x \cdot x$, we know $p \mid x$.

p prime
If $p \mid ab$, then $p \mid a$ or $p \mid b$

So, $x = pk$ for some $k \in \mathbb{Z}$.

Plug this back into $(*)$ to get:

$$py^2 = (pk)^2$$

So, $py^2 = p^2 k^2$

So, $y^2 = pk^2$

Thus, $p \mid y^2$.

Since p is prime we have $p \mid y$.

So, $p \mid x$ and $p \mid y$.

Thus, $\gcd(x,y) \geq p > 1$.

This contradicts $\gcd(x,y)=1$.

Hence $\sqrt{p}$ is irrational. $\blacksquare$