Math 4460
3/22/23

Summary:

There are three kinds of Pythagorean triples.

① $(x, 0, \pm x)$

$$x^2 + 0^2 = (\pm x)^2$$

Ex:
$(x,y,z) = (3,0,3)$

② $(0, y, \pm y)$

$$0^2 + y^2 = (\pm y)^2$$

Ex:
$(x,y,z) = (0,5,-5)$

③ the ones that are multiples of positive, primitive Pythagorean triples with possible sign adjustments.

$\boxed{(5,12,13) \text{ positive primitive}} \xrightarrow{\times 2} \boxed{(10,24,26)} \xrightarrow{\text{sign}} \boxed{(-10,24,26)}$

$\boxed{\text{Let } (x,y,z) \text{ be a positive, primitive, Pythagorean triple.}}$

$$\boxed{\begin{array}{l}\text{So, } x>0,\ y>0,\ z>0, \\ \gcd(x,y,z)=1, \\ x^2+y^2=z^2. \\ x,y,z\in\mathbb{Z}\end{array}}$$

- Let's show x and y can't both be even. Why?

Suppose x and y are both even.
Then in $\mathbb{Z}_2=\{\overline{0},\overline{1}\}$ we would have $\overline{x}=\overline{0}$ and $\overline{y}=\overline{0}$.

Then,

$$\overline{z}^2=\overline{z^2}=\overline{x^2+y^2}$$

$$= \overline{x^2} + \overline{y^2}$$

$$= \overline{x}^2 + \overline{y}^2$$

$$= \overline{0}^2 + \overline{0}^2 = \overline{0}.$$

So, $\overline{z}^2 = \overline{0}$.

Then $\overline{z} = \overline{0}$

$\boxed{\text{if } \overline{z} = \overline{1} \text{ then } \overline{z}^2 = \overline{1}}$

But then z would be even.

But then x, y, z would be even and then $2 | x$, $2 | y$, $2 | z$.

So, $\gcd(x, y, z) \geq 2$

Contradiction.

So, x and y can't both be even.

- x and y can't both be odd.

Why?

Suppose x and y are both odd.

Recall in $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$

a is odd iff $\bar{a} = \bar{1}$ or $\bar{a} = \bar{3}$

And in $\mathbb{Z}_4$, $\bar{1}^2 = \bar{1}$

and $\bar{3}^2 = \bar{9} = \bar{1}$.

Thus, if x and y are both odd then $\bar{x}^2 = \bar{1}$ and $\bar{y}^2 = \bar{1}$.

So, $\bar{z}^2 = \bar{x}^2 + \bar{y}^2 = \bar{1} + \bar{1} = \bar{2}$

But this can't happen in $\mathbb{Z}_4$ by this table

$\bar{z}$	$\bar{z}^2$
$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{1}$
$\bar{2}$	$\bar{4} = \bar{0}$

$\bar{3} \mid \bar{9} = \bar{1}$

Thus, x and y can't both be odd.

Therefore, either

x is odd and y is even

or

x is even and y is odd.

Since our equation $x^2 + y^2 = z^2$ is symmetric in x and y we can just solve one of these cases. We will find all solutions where x is odd and y is even.

Let us assume that

x is odd and y is even

Then, in $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ we have

$$\bar{z}^2 = \bar{x}^2 + \bar{y}^2$$
$$= \bar{1}^2 + \bar{0}^2$$
$$= \bar{1}$$

So, $\bar{z}^2 = \bar{1}$ in $\mathbb{Z}_2$.

Thus, $\bar{z} = \bar{1}$ in $\mathbb{Z}_2$.

Hence z is odd.

Since x is odd and z is odd
we know $z-x$ is even
and $z+x$ is even.

Since $x^2+y^2=z^2$ we have

$$y^2 = z^2 - x^2$$

Thus,

$$y^2 = (z+x)(z-x)$$

So, dividing by 4 gives:

$$\boxed{\left(\frac{y}{2}\right)^2 = \left(\frac{z+x}{2}\right)\left(\frac{z-x}{2}\right)} \quad (*)$$

Note: $\frac{y}{2}, \frac{z+x}{2}, \frac{z-x}{2} \in \mathbb{Z}$ since
$y, z+x, z-x$ are all even.

We want to show $\gcd\left(\frac{z+x}{2}, \frac{z-x}{2}\right)=1.$

Any common divisor of $\frac{z+x}{2}$ and $\frac{z-x}{2}$ must also divide

their sum

$$\frac{z+x}{2} + \frac{z-x}{2} = z$$

and their difference

$$\frac{z+x}{2} - \frac{z-x}{2} = x.$$

So, if we can show that $\gcd(x,z)=1$, then this will imply that $\gcd\left(\frac{z+x}{2}, \frac{z-x}{2}\right)=1$

Why is $\gcd(x,z)=1$?

$\boxed{\begin{array}{l}\underline{\text{HW 3-5(a)}} \\ \gcd(x,z)\neq 1 \text{ iff there} \\ \text{exists a prime } p \text{ where} \\ p \mid x \text{ and } p \mid z\end{array}}$

Suppose $\gcd(x,z)\neq 1$.

Then by HW, there exists a prime p where $p \mid x$ and $p \mid z$.

Then, $p \mid x^2$ and $p \mid z^2$.

So, $p \mid (z^2-x^2)$.

$y^2 = z^2 - x^2$

Then $p \mid y^2$

Since p is prime and $p \mid y \cdot y$ we know $p \mid y$.

p prime $p \mid ab$ implies $p \mid a$ or $p \mid b$

But then $p \mid x$, $p \mid y$, $p \mid z$ and
$\gcd(x, y, z) \geq p$.
Contradiction since $\gcd(x, y, z) = 1$.

Thus, $\gcd(x, z) = 1$
and so $\boxed{\gcd\left(\frac{z+x}{2}, \frac{z-x}{2}\right) = 1.}$

Recall this theorem: If A, B, C are positive integers and $\gcd(A, B) = 1$ and $C^n = AB$ then there exist positive integers R, S where $A = R^n$ and $B = S^n$ | topic 3 section

Our situation from (*) is that

$$\left(\frac{y}{2}\right)^2 = \left(\frac{z+x}{2}\right)\left(\frac{z-x}{2}\right)$$

with $\gcd\left(\frac{z+x}{2}, \frac{z-x}{2}\right) = 1$

thm above $C^2 = AB$

Hence, $\frac{z+x}{2} = r^2$ and $\frac{z-x}{2} = s^2$

where r, s are positive integers and $\underbrace{\gcd(r,s) = 1}$

because $\gcd\left(\frac{z+x}{2}, \frac{z-x}{2}\right) = 1$

So, $\left(\frac{y}{2}\right)^2 = \left(\frac{z+x}{2}\right)\left(\frac{z-x}{2}\right) = r^2 s^2$

y, r, s are positive.

So, $\frac{y}{2} = rs$.

Or, $\boxed{y = 2rs}$

Note that $r^2 = \frac{z+x}{2} > \frac{z-x}{2} = s^2$.

So, $\boxed{r > s}$

$\bar{z} = \bar{1}$ in $\mathbb{Z}_2$

Also, since z is odd and

$$z = \frac{z+x}{2} + \frac{z-x}{2} = r^2 + s^2$$

we must have that r and s have opposite parity.

That is, either

r is odd & s is even

In $\mathbb{Z}_2$

$\bar{z}$

$\bar{r}$	$\bar{s}$	$\bar{r}^2 + \bar{s}^2$
$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{0}$	$\bar{1}$	$\bar{1}$
$\bar{1}$	$\bar{0}$	$\bar{1}$
$\bar{1}$	$\bar{1}$	$\bar{0}$

or r is even & s is odd.

Theorem: If (x,y,z) is a positive, primitive Pythagorean triple, with y even, then

$$x = r^2 - s^2 \quad \leftarrow \quad \boxed{x = \frac{z+x}{2} - \frac{z-x}{2}}$$

$$y = 2rs$$

$$z = r^2 + s^2$$

Where r and s are positive integers of opposite parity and $r > s > 0$ and $\gcd(r,s) = 1$

s	r	$x = r^2 - s^2$	$y = 2rs$	$z = r^2 + s^2$
1	2	3	4	5
1	4	15	8	17
1	6	35	12	37
2	3	5	12	13
2	5	21	20	29
3	4	7	24	25
⋮	⋮	⋮	⋮	⋮