Math 4460
4/24/23

HW 4

#12 Show that $4 \nmid (n^2+2)$ for any integer n.

<u>proof:</u> Suppose there exists an integer n where $n^2+2 = 4\ell$ for some $\ell \in \mathbb{Z}$.

Suppose $4 \mid (n^2+2)$

Then in $\mathbb{Z}_4$ we would have

$$\overline{n}^2 + \overline{2} = \overline{0}$$

since $\overline{4} = \overline{0}$ in $\mathbb{Z}_4$

Let's show this can't happen.

$\overline{n}$	$\overline{n}^2+\overline{2}$
$\overline{0}$	$\overline{2}$
$\overline{1}$	$\overline{3}$
$\overline{2}$	$\overline{6} = \overline{2}$
$\overline{3}$	$\overline{11} = \overline{3}$

From the table we see that there is no $\overline{n} \in \mathbb{Z}_4$ where $\overline{n}^2 + \overline{2} = \overline{0}$.

So we get a contradiction.

Thus, $4 \nmid (n^2+2)$ for all n. ∎

HW 4

(14) Prove that $x^2 - 5y^2 = 2$ has no integer solutions.

Proof by contradiction: Suppose there did exist integers x and y where $x^2 - 5y^2 = 2$.

$\overline{5} = \overline{0}$

Then in $\mathbb{Z}_5$ we would have $\overline{x}^2 = \overline{2}$

$\overline{x}$	$\overline{x}^2$
$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{1}$
$\overline{2}$	$\overline{2}^2 = \overline{4}$
$\overline{3}$	$\overline{3}^2 = \overline{9} = \overline{4}$
$\overline{4}$	$\overline{4}^2 = \overline{16} = \overline{1}$

We see from the table that there is no $\overline{x} \in \mathbb{Z}_5$ where $\overline{x}^2 = \overline{2}$. Contradiction.

Hence, $x^2 - 5y^2 = 2$ does not have integer solutions. ▨

HW 3

(3) Prove that $\log_{10}(2)$ is irrational.

proof by contradiction:

Suppose $\log_{10}(2)$ is rational.

Then $\log_{10}(2) = \frac{x}{y}$ where x and y are positive integers and $\gcd(x,y)=1$.

$[\log_{10}(t) > 0$ iff $t>1]$

Thus, $10^{x/y} = 2$.

So, $10^x = 2^y$.

So, $\boxed{2^x 5^x = 2^y}$.

Since prime factorization is unique and there are no 5's on the right side of the equation we must have that $x=0$.

Then we get $2^0 5^0 = 2^y$.

Thus $1 = 2^y$

But then $y=0$.

Contradiction.

Thus, $\log_{10}(2)$ is irrational. $\blacksquare$

$\boxed{\text{HW 3}}$

5 (b)

Let a, b, n be positive integers.
Prove $\gcd(a,b)>1$ iff $\gcd(a^n, b^n)>1$

<u>proof:</u>

($\Rightarrow$) Suppose $d=\gcd(a,b)>1$.

Then, $d \mid a$ and $d \mid b$.

So, $a=dk$ and $b=d\ell$, where $k, \ell \in \mathbb{Z}$.

$\downarrow \times a^{n-1}$ $\qquad$ $\downarrow \times b^{n-1}$

Then, $a^n = d(ka^{n-1})$ and $b^n = d(\ell b^{n-1})$.

So, $d \mid a^n$ and $d \mid b^n$.

Thus, $\gcd(a^n, b^n) \geq d > 1$.

($\Leftarrow$) Suppose $d = \gcd(a^n, b^n) > 1$.

Since $d > 1$ we know there exists a prime p where $p \mid d$.

Since $d = \gcd(a^n, b^n)$ we know $d \mid a^n$ and $d \mid b^n$.

Since $p \mid d$ and $d \mid a^n$, we know $p \mid a^n$.
Since $p \mid d$ and $d \mid b^n$ we know $p \mid b^n$.

Since p is prime and $p \mid \underbrace{a \cdot a \cdots a}_{a^n}$, then $\boxed{p \mid a.}$

Since p is prime and $p \mid \underbrace{b \cdot b \cdots b}_{b^n}$, then $\boxed{p \mid b.}$

Thus, $\gcd(a,b) \geq p > 1$

$\boxed{p \text{ is prime}}$

$\blacksquare$

$\boxed{\text{HW 5}}$

① $\mathbb{Z}_7 = \{\cancel{\overline{0}}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$

gcd(0,7) = 7
gcd(1,7) = 1
gcd(2,7) = 1
gcd(3,7) = 1
gcd(4,7) = 1
gcd(5,7) = 1
gcd(6,7) = 1

$\mathbb{Z}_7^{\times} = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$

$\overline{1} \cdot \overline{1} = \overline{1} \leftarrow \boxed{\overline{1}^{-1} = \overline{1}}$

$\overline{2} \cdot \overline{4} = \overline{8} = \overline{1} \leftarrow \boxed{\overline{2}^{-1} = \overline{4}}$ and $\boxed{\overline{4}^{-1} = \overline{2}}$

$\overline{3} \cdot \overline{5} = \overline{15} = \overline{1} \leftarrow \boxed{\overline{3}^{-1} = \overline{5}}$ and $\boxed{\overline{5}^{-1} = \overline{3}}$

$\overline{6} \cdot \overline{6} = \overline{36} = \overline{1} \leftarrow \boxed{\overline{6}^{-1} = \overline{6}}$