4460 Test 2 Solutions

$$\begin{array}{r} 335 \\ 3\overline{)1005} \\ -9 \\ \hline 10 \\ -9 \\ \hline 15 \\ -15 \\ \hline 0 \end{array}$$

① (a) $4-(-1001) = 1005 = 3 \cdot 335$

Since $3 \mid (4-(-1001))$ we know that $4 \equiv -1001 \pmod 3$

TRUE

(b) $68-(-3) = 71$

$12 \nmid 71$

$$\begin{array}{r} 5 \\ 12\overline{)71} \\ -60 \\ \hline 11 \end{array}$$

remainder

Thus, $\overline{-3} \neq \overline{68}$

So the statement $\overline{-3} = \overline{68}$ is false

② (a) $\overline{3} \cdot \overline{5} + \overline{2}^3 + \overline{4} \cdot \overline{4} \cdot \overline{-3} = \overline{15} + \overline{8} + \overline{16} \cdot \overline{-3}$

$= \overline{3} + \overline{2} + \overline{4} \cdot \overline{3}$

$= \overline{5} + \overline{12}$

$= \overline{5} + \overline{0} = \overline{5}$

② (b) $\overline{-14}^4 = \overline{-2}^4 = \overline{16} = \overline{4}$

$\overline{-14} = \overline{-2}$ in $\mathbb{Z}_6$

$\overline{16} = \overline{4}$ in $\mathbb{Z}_6$

3 (a)

$\mathbb{Z}_{20}^{\times} = \{\bar{1}, \bar{3}, \bar{7}, \bar{9}, \overline{11}, \overline{13}, \overline{17}, \overline{19}\}$

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3 (b) $\bar{9} \cdot \overline{11} = \overline{99} = \bar{1}$

$$\begin{array}{r} 7 \\ 14 \overline{)\,99} \\ -98 \\ \hline 1 \end{array}$$

So, $\bar{9}^{-1} = \overline{11}$

A HW 4 - Problem #10

B HW 3 - Problem #6

(c) Suppose $\sqrt{\frac{3}{5}}$ is rational.

Then $\sqrt{\frac{3}{5}} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b \neq 0$, and $\gcd(a,b)=1$.

Thus, $\frac{3}{5} = \frac{a^2}{b^2}$.

So, $3b^2 = 5a^2$.

Then $3 \mid 5a^2$.

Since 3 is prime $3 \mid 5$ or $3 \mid a^2$.

Since $3 \nmid 5$ we know $3 \mid a^2$.

Since 3 is prime and $3 \mid a \cdot a$ we know $3 \mid a$.

Thus $a = 3k$ where $k \in \mathbb{Z}$.

So, $3b^2 = 5(3k)^2$.

Thus, $3b^2 = 5 \cdot 3^2 \cdot k^2$.

So, $b^2 = 3 \cdot 5 \cdot k^2$.

Thus, $3 \mid b^2$.

Since 3 is prime and $3 \mid b \cdot b$, we know $3 \mid b$.

But then $3 \mid a$ and $3 \mid b$ which contradicts $\gcd(a,b)=1$.

Therefore, $\sqrt{3/5}$ is irrational. $\square$

(P) Suppose by way of contradiction that there exist integers x, y with $15x^2 - 10x + 7y^2 = 11$.

Then in $\mathbb{Z}_5$ we would have

$$\overline{15}\,\overline{x}^2 + \overline{(-10)}\,\overline{x} + \overline{7}\,\overline{y}^2 = \overline{11}$$

$\overline{15} = 0$
$\overline{-10} = 0$
in $\mathbb{Z}_5$

So, $\overline{7}\,\overline{y}^2 = \overline{11}$

$\overline{7} = \overline{2}$ in $\mathbb{Z}_5$
$\overline{11} = \overline{1}$

Thus, $\overline{2}\,\overline{y}^2 = \overline{1}$.

So, $\overline{3} \cdot \overline{2}\,\overline{y}^2 = \overline{3}$.

$\overline{3} \cdot \overline{2} = \overline{6} = \overline{1}$ in $\mathbb{Z}_5$

Thus, $\overline{y}^2 = \overline{3}$.

However in $\mathbb{Z}_5$ we have $\overline{0}^2 = \overline{0}$, $\overline{1}^2 = \overline{1}$, $\overline{2}^2 = \overline{4}$, $\overline{3}^2 = \overline{9} = \overline{4}$, $\overline{4}^2 = \overline{16} = \overline{1}$

Thus, there is no $\overline{y} \in \mathbb{Z}_5$ with $\overline{y}^2 = \overline{3}$.

Contradiction.

Thus, $15x^2 - 10x + 7y^2 = 11$ has no integer solutions.