9/14 P.1

Monday Week 13, November 14, 2016

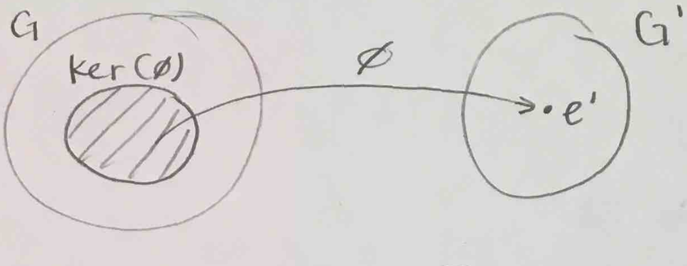
[redacted]

continued...

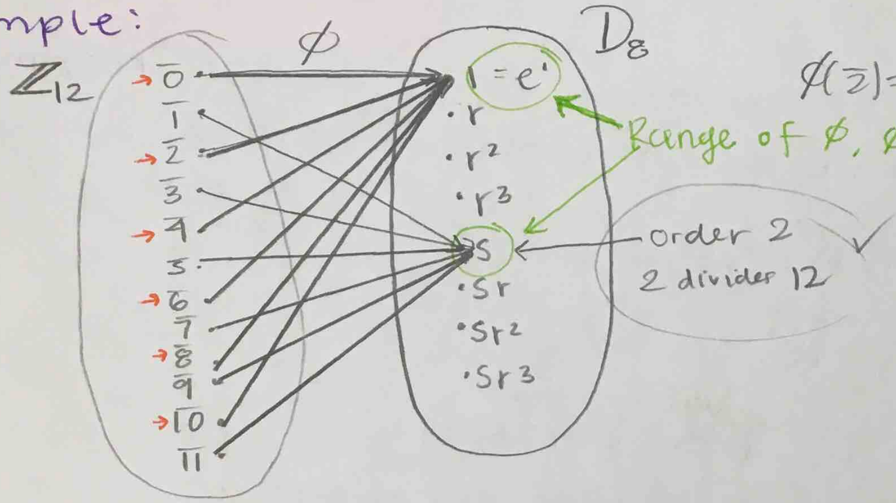
Def: Let $\phi : G \to G'$ be a group homomorphism. Let e' be the identity of G'

The **Kernel** of ϕ is:

$$\ker(\phi) = \{x \in G \mid \phi(x) = e'\}$$



Example:



$\phi(\bar{2}) = \phi(\bar{1} + \bar{1}) = \phi(\bar{1}) * \phi(\bar{1}) = s \cdot s = s^2 = 1$

Range of ϕ, $\phi(G) = \{1, s\}$

order 2 ✓
2 divider 12

$\phi(\{\bar{0}, \bar{1}\}) = \{1, s\}$

$\phi(\{\bar{0}, \bar{2}\}) = \{1\}$

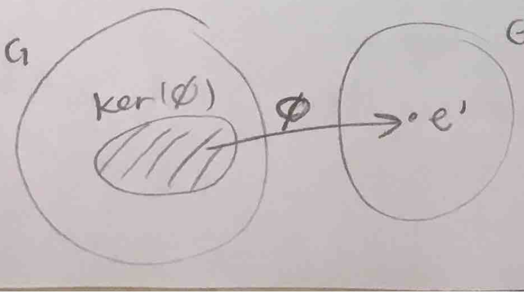
$\ker(\phi) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}\} = \langle \bar{2} \rangle$

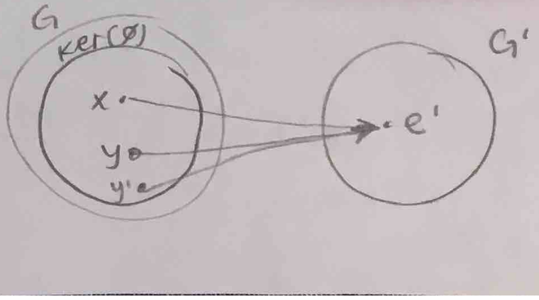
Theorem: Let $\phi : G \to G'$ be a homomorphism of groups. Then $\ker(\phi) \leq G$

Proof: Let e and e' be the identity element of G and G' respectively.

① From a thm. in class we have that $\phi(e) = e'$ so $e \in \ker(\phi)$

$H \leq G$ iff
① $e \in H$
② $xy^{-1} \in H$ $\forall x, y \in H$





② Let $x, y \in \ker(\phi)$. so, $\phi(x) = e'$ and $\phi(y) = e'$

By a thm in class.

$$\phi(y^{-1}) = [\phi(y)]^{-1} = [e']^{-1} = e'$$

don't need this

so, $y^{-1} \in \ker(\phi)$.

so, $\phi(xy^{-1}) = \phi(x)\phi(y^{-1}) = \phi(x)[\phi(y)]^{-1}$

$= (e')(e')^{-1} = e'e' = e'$

Thus $xy^{-1} \in \ker(\phi)$

therefore, $\ker(\phi) \leq G$ ⊠

ker(∅)
normal subgrups
G/N ← set of left cosets
becomes a group iff N is normal

Def: A subgroup H of a group G is called normal if $gH = Hg$ $\forall g \in G$. If H is normal in G then we write $H \trianglelefteq G$.

Example: $G = D_6 = \{1, r, r^2, s, sr, sr^2\}$

$H = \langle r \rangle = \{1, r, r^2\}$, is H normal?

equal

left cosets

$1 \cdot H = \{1, r, r^2\}$

$r \cdot H = \{r, r^2, r^3 = 1\}$

$r^2 \cdot H = \{r^2, 1, r\}$

equal

$sH = \{s, sr, sr^2\}$

$srH = \{sr, sr^2, sr^3 = s\}$

$sr^2H = \{sr^2, s, sr\}$

Facts: $aH = bH$ iff $a \in bH$ iff $a^{-1}b \in H$

$Ha = Hb$ iff $a \in Hb$ iff $ba^{-1} \in H$

Right cosets

equal

$H = \{1, r, r^2\}$

$Hr = \{1, r, r^2\}$

$Hr^2 = \{1, r, r^2\}$

equal

Hs $\{s, r \cdot s, r^2 \cdot s\} = \{s, sr^2, sr\}$

$Hsr = \{s, sr, sr^2\}$

$Hsr^2 = \{s, sr, sr^2\}$

If we compare we can observe that

$1H = H1$, $rH = Hr$, $r^2H = Hr^2$, $sH = Hs$, $srH = Hsr$, $sr^2H = Hsr^2$ ✓

∴ H is normal.

9/14 P.2

Example: $G = D_6 = \{1, r, r^2, s, sr, sr^2\}$

$H = \langle s \rangle = \{1, s\}$

left coset

right coset

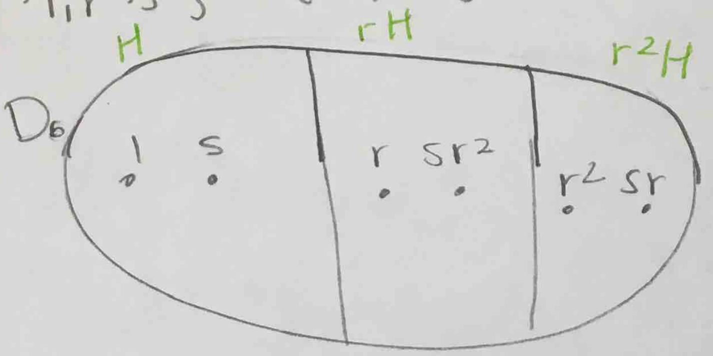
$H = \{1, s\} = sH$

$H = \{1, r\} = Hs$

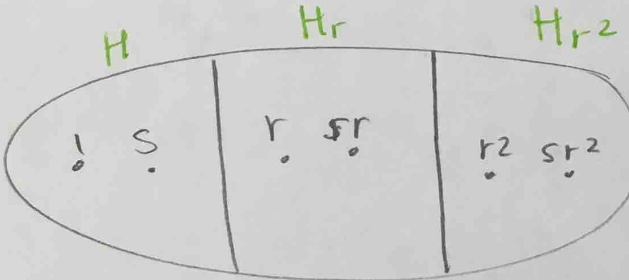
$rH = \{r, rs\} = \{r, sr^2\} = sr^2H \xleftrightarrow{\text{not equal}} Hr\{1\cdot r, sr\} = \{r, sr\} = Hsr$

$r^2H = \{r^2\cdot 1, r^2\cdot s\} = \{r^2, sr\} = srH \leftrightarrow Hr^2 = \{s2, sr^2 = Hsr^2$

left cosets

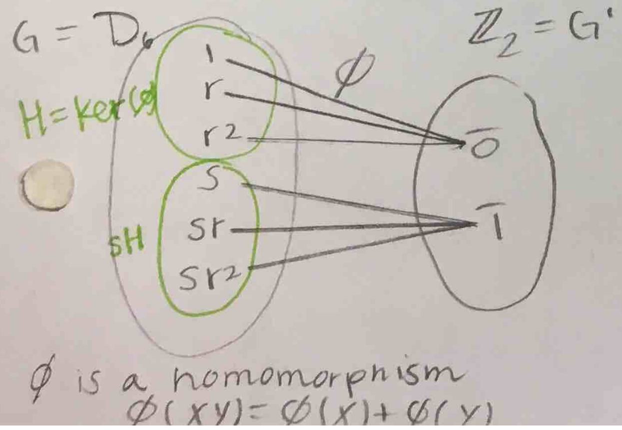


Right cosets



This H is NOT normal since for example $rH \neq Hr$

Theorem: Let $\phi: G \to G'$ be a group homomorphism then $\ker(\phi) \trianglelefteq G$



ϕ is a homomorphism $\phi(xy) = \phi(x) + \phi(y)$

proof →

ϕ is a homomorphism $\phi(xy) = \phi(x) + \phi(y)$

Proof

Let $H = \ker(\phi)$. we already showed that $H \leq G$.

lets now show H is normal.

It turns out that $\forall g \in G$ we have that

$$\underset{\uparrow}{gH =} \{x \in G \mid \phi(x) = \phi(g)\} \underset{\uparrow}{=} Hg$$

we'll show this

This is similar (HW)

let $g \in G$ be fixed

$gH \supseteq \{x \in G \mid \phi(x) = \phi(g)\}$

let $x \in G$ where $\phi(x) = \phi(g)$

so, $\phi(g)^{-1}\phi(x) = \phi(g^{-1}\phi(g))$

then $\phi(g)^{-1}\phi(x) = e'$

so, $\phi(g^{-1}x) = e'$

so $g^{-1}x \in \ker(\phi) = H$

so, $g^{-1}x = h$ where $h \in H$ so $x = gh$

thus, $x \in gH$

scratchwork

goal $x \in gH$

$x = gh$, $h \in H$

$g^{-1}x = h$

$gH \subseteq \{x \in G \mid \phi(x) = \phi(g)\}$

Proof: Let $x \in gH$

then $x = gh$ where $h \in H$

since $h \in H = \ker(\phi)$, we know $\phi(h) = e'$

so $\phi(x) = \phi(gh) = \phi(g)\phi(h) = \phi(g)e' = \phi(g)$

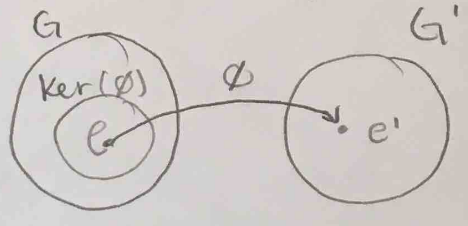
so, $x \in \{x \in G \mid \phi(x) = \phi(g)\}$ ☒

11/16 P.1 Wednesday Week 13 November 16, 2016

Theorem: Let $\phi: G \to G'$ be a group homomorphism

ϕ is 1-1 iff $\ker(\phi) = \{e\}$

[here e and e' are the identity elements of G and G'.]



Proof of theorem:

($\Rightarrow$) Suppose ϕ is 1-1

we know $\phi(e) = e'$ so, $e \in \ker(\phi)$

Then $\phi(x) = e'$ so, $\phi(x) = \phi(e)$

since ϕ is 1-1 then $x = e$

so $\ker(\phi) = \{e\}$.

($\Leftarrow$) Suppose $\ker(\phi) = \{e\}$

Suppose $\phi(x) = = \phi(y)$

where $x, y \in G$. So, $[\phi(y)]^{-1}\phi(x) = [\phi(y)]^{-1}\phi(y)$

then $\phi(y')\phi(x) = e'$ so $\phi(y^{-1}x) = e'$

thus $y'x \in \ker(\phi)$ so $y'x = e$, Thus $yy^{-1}x = ye$

so $x = y$ Thus ϕ is 1-1. $\boxtimes$

the size of the ker ϕ is 2 and so is every other element this is a 2-1 function

$$\left.\begin{aligned} H &= \{\bar{0}, \bar{3}\} \\ \bar{1}+H &= \{\bar{1}, \bar{4}\} \\ \bar{2}+H &= \{\bar{2}, \bar{5}\} \end{aligned}\right\} \text{left cosets of } H$$

HW #9

Factor Groups:

Idea:

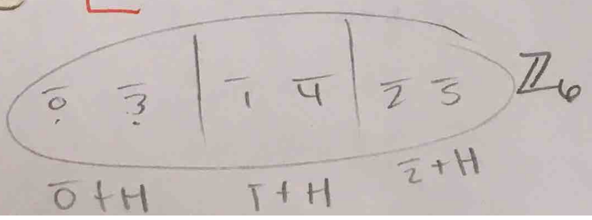
$\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

3 left cosets of H

$H = \{\bar{0}, \bar{3}\} = \bar{0} + H = \bar{3} + H$

$\{\bar{1}, \bar{4}\} = \bar{1} + H = \bar{4} + H$

$\{\bar{2}, \bar{5}\} = \bar{2} + H = \bar{5} + H$



lets make a way to add cosets if we try this:

$(\bar{a}+H) + (\bar{b}+H) = (\bar{a}+\bar{b}) + H$

Example

$(\bar{1}+H) + (\bar{2}+H) = (\bar{1}+\bar{2}) + H =$

$\bar{3} + H = \bar{0} + H$

Theorem: Let G be a group and H be a subgroup of G. Define the following operation on the left cosets of H:

$$(aH)(bH) = (ab)H \text{ , where } a,b \in G$$

This operation is well defined iff H is a normal subgroup of G.

[Well defined means:
If $aH = cH$ and $bH = dH$ where $a,b,c,d \in G$ then
$(aH)(bH) = (cH)(dH)$]

Proof: (we will just proof $\Leftarrow$)
Suppose H is a normal subgroup of G.
This means $gH = Hg$ $\forall g \in G$.
Suppose $aH = cH$ and $bH = dH$ where $a,b,c,d \in G$.
Since $aH = cH$ then $a \in cH$ since $bH = dH$ then $b \in dH$
so, $a = ch$ where $h \in H$ so, $b = dh_2$ where $h_2 \in H$
then, $ab = ch_1dh_2$ * note that $h_1d \in Hd$ but since H is normal, $Hd = dh_3$ where $h_3 \in H$
so, $hd = dh_3$ where $h_3 \in H$
Therefore, $ab = ch_1dh_2 = cdh_3h_2$. so $ab \in cdH$
Ergo $abH = cdH$, so $(aH)(bH) = (cH)(dH)$ ▨

Corollary: Let G be a group and H be a normal subgroup. Denote the set of left cosets by G/H
G/H is a group under the operation. say: "G mod H"

$$(aH)(bH) = (ab.H)$$

The identity element is $eH = H$ where e is the identity of G

The inverse of aH is $(a^{-1})H$.

11/16 P.2

Proof:

Closure: Let $a, b \in G$ then since G is a group $ab \in G$ so, $(aH)(bH) = (ab)H$ is a left coset.

Associativity Let $a, b, c \in G$ then

$$aH[(bH)(cH)]$$
$$= a(bc)H$$

since G is associative $\} \rightarrow$
$$= (ab)cH = [(ab)H]cH$$
$$= [(aH)(bH)]cH$$

Identity: Let $a \in G$

then $(eH)(aH) = (ea)H = aH$ and
$(aH)(eH) = (ae)H = aH$

so, $eH = H$ is an identity.

Inverses

Let $a \in G$

then $[(a^{-1})H][aH] = (a^{-1}a)H = eH$
$(aH)(a^{-1}H) = (aa^{-1})H = eH$

so, $a^{-1}H$ is an inverse for aH ▨

Def: If $H \trianglelefteq G$

The set of left-cosets, denoted by G/H, is called the factor group of G by H or (Quotient group)