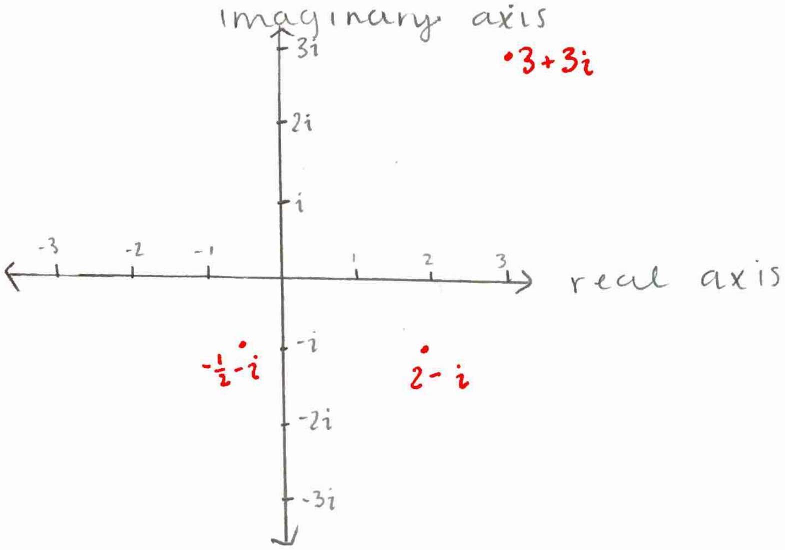
Week 2 Monday aug. 29. 2016

Complex Numbers

$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$



$i^2 = -1$

$i = \sqrt{-1}$

addition

$(3+2i) + (10-i) = 13 + i$

multiplication

$(3+2i)(10-i) = 30 - 3i + 20i - 2\underbrace{i^2}_{=-1}$

$= 32 + 17i$

division

$\frac{1+3i}{1+5i} = \frac{1+3i}{1+5i} \cdot \frac{1-5i}{1-5i}$ ← mult by the conjugate

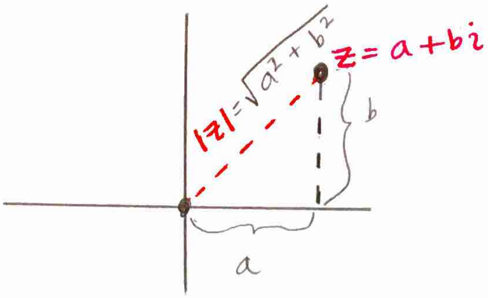
$= \frac{1 - 2i - 15\underbrace{i^2}_{=-1}}{1 - 25\underbrace{i^2}_{=-1}} = \frac{16 - 2i}{26} = \frac{16}{26} - \frac{2}{26}i$

$= \frac{8}{13} - \frac{1}{13}i$

Def: Let $z = a + bi \in \mathbb{C}$

The **conjugate** of z is $\bar{z} = a - bi$

The **norm** of z is $|z| = \sqrt{a^2 + b^2}$



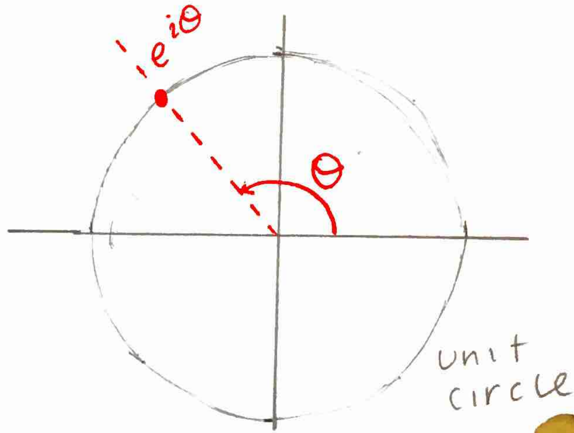
ex: $|1 - 5i| = \sqrt{1^2 + (-5)^2} = \sqrt{26}$

Euler's Formula

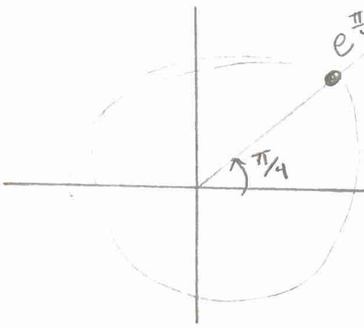
Let θ be a real number

then:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$



Example

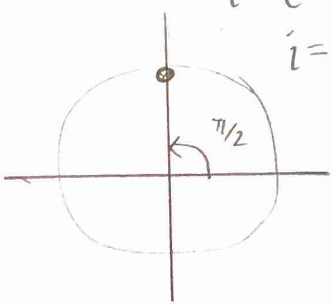


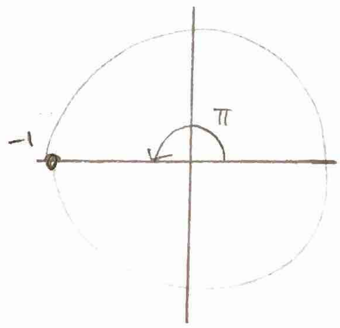
$$e^{\frac{\pi}{4}i} = \cos(\pi/4) + i\sin(\pi/4)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$i = e^{i\theta}$

$i = e^{i(\pi/2)}$





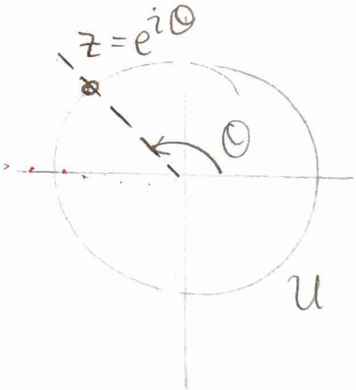
$-1 = e^{i\theta}$

$$\boxed{-1 = e^{i\pi}}$$

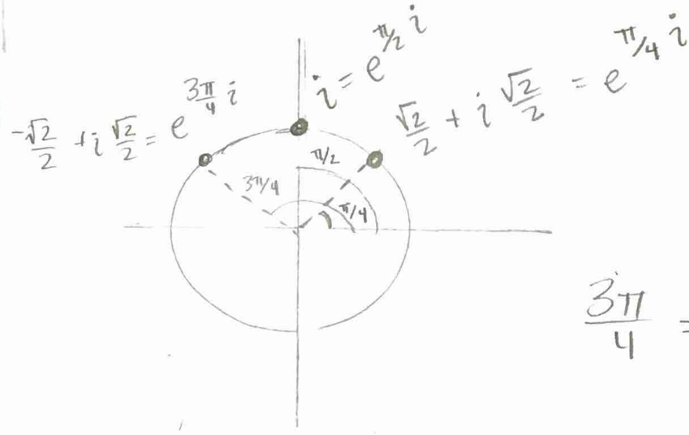
8/30 P. 2

- The **Unit Circle** is

$$U = \{ z \in \mathbb{C} \mid |z| = 1 \} = \{ e^{i\theta} \mid \theta \in \mathbb{R} \}$$



Example:



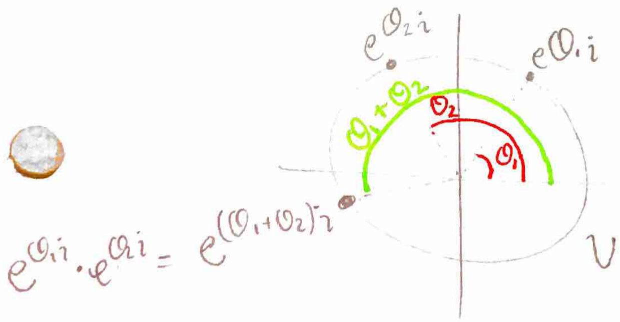
$$\frac{3\pi}{4} = \frac{\pi}{2} + \frac{\pi}{4}$$

multiply them

$$i \cdot \left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \right] = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

In general, if θ_1 and θ_2 are real then

$$e^{\theta_1 i} \cdot e^{\theta_2 i} = e^{[\theta_1 + \theta_2] i}$$



$\boxed{U \text{ is closed under multiplication}}$

Fact Given $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, let $\theta, \theta_1, \theta_2 \in \mathbb{R}$

then $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$

Proof we have that

$$e^{i\theta} \cdot e^{i\theta_2} = [\cos(\theta_1) + i\sin(\theta_1)]\,[\cos(\theta_2) + i\sin(\theta_2)]$$

$$= [\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)]$$

$$= + i\,[\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)]$$

$\cos(A+B) =$
$\cos(A)\cos(B) - \sin(A)\sin B$

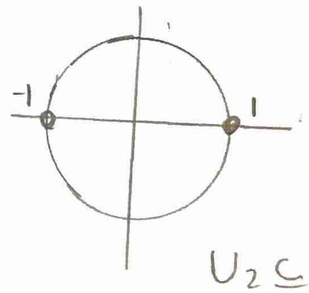
$\sin(A+B) =$
$\cos(A)\sin(B) + \cos(B)\sin(A)$

$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$$

$$= e^{i(\theta_1 + \theta_2)}$$

Def The set of **n^{th} roots of unity** is

$$U_n = \{z \in \mathbb{C} \mid z^n = 1\}$$



Ex $U_2 = \{z \in \mathbb{C} \mid z^2 = 1\} = \{1, -1\}$ $\leftarrow U_2$

$U_2 \subseteq U$

$U_4 = \{z \in \mathbb{C} \mid z^4 = 1\} = \{1, -1, i, -i\}$

$\left.\begin{array}{l} -1^1 = -1 \\ -1^2 = 1 \end{array}\right\}$ all 2 angles evenly spaced

$z^4 - 1 = 0$

$(z^2 - 1)(z^2 + 1) = 0$

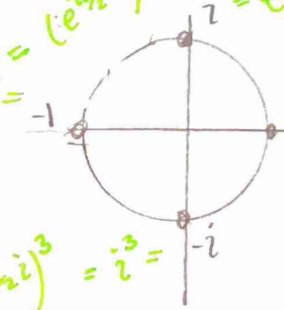
$(z-1)(z+1)(z-i)(z+i) = 0$

$i = e^{\frac{2\pi}{4}i} = e^{\frac{\pi}{2}i}$

$i^2 = (e^{\frac{\pi}{2}i})^2$

$1 = i^0 = (e^{\pi/2\, i})^0$

$(e^{\pi/2\, i})^3 = i^3 =$



U_4

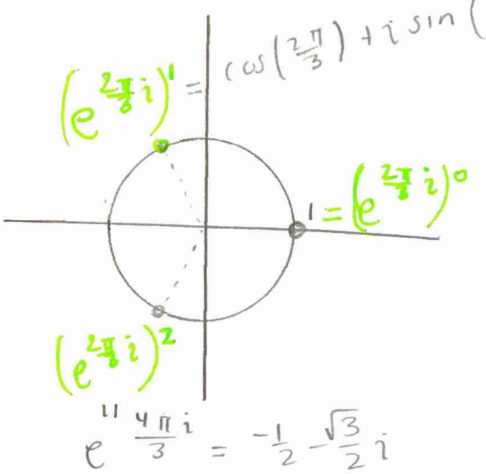
$\left.\begin{array}{l} i^1 = i \\ i^2 = -1 \\ i^3 = -i \\ i^4 = 1 \end{array}\right\}$ all 4 angles evenly spaced

8/30 P. 3

In general,

$$U_n = \left\{ \left(e^{\frac{2\pi}{n} i}\right)^k \;\middle|\; k = 0, 1, 2, \ldots, n-1 \right\}$$

Example calculate U_3



$$U_3 = \left\{ \left(e^{\frac{2\pi}{3} i}\right)^k \;\middle|\; k = 0, 1, 2 \right\}$$

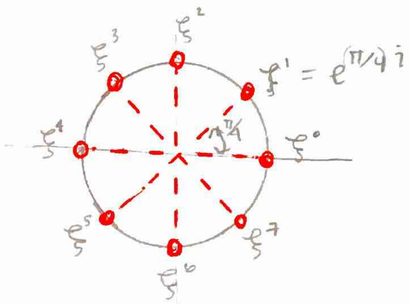
$$= \left\{ \left(e^{\frac{2\pi}{3} i}\right)^0, \left(e^{\frac{2\pi}{3} i}\right)^1, \left(e^{\frac{2\pi}{3} i}\right)^2 \right\}$$

$$= \left\{ 1, e^{\frac{2\pi}{3} i}, e^{\frac{4\pi i}{3}} \right\}$$

$$= \left\{ 1, \xi, \xi^2 \right\} \quad \left(\xi = e^{\frac{2\pi i}{3}}\right)$$

Example calculate $U_8 = \left\{ \xi^k \;\middle|\; k = 0, 1, 2, 3, 4, 5, 6, 7 \right\}$

where $\xi = e^{\frac{2\pi}{8} i} = e^{\pi/4\, i}$



$$\left[e^{\frac{2\pi i}{n} k}\right]^n = e^{2\pi i k} = \cos(2\pi k) + \underbrace{i\sin(2\pi k)}_{0} = 1$$

there are n things in Un

$2^n - 1$

Groups A group $(G,*)$ consits of a nonempty set G and a binary operation $*$ on G where

(1) If $a,b \in G$, then $a*b \in G$ (closure)

(2) If $a,b,c \in G$, then $a*(b*c)=(a*b)*c$ (associativity)

(3) $\exists$ an element $e \in G$ s.t. $a*e=e*a=a$ $\forall a \in G$ (identity)

(4) given $a \in G$ $\exists$ $b \in G$ where $a*b=b*a=e$ inverse

Example: $(\mathbb{Z},+)$ is a group recall $\mathbb{Z}_n=\{\bar{0},\bar{1},\bar{2},\dots,\overline{n-1}\}$

① ② let $a,b,c \in \mathbb{Z}$

Then $a+b \in \mathbb{Z}$ then $a+(b+c)=(a+b)+c$

③ $e=0$ works s.t. $a+0=0+a=a$ $\forall a \in \mathbb{Z}$

④ $b=(-a)$ works s.t $a+(-a)=(-a)+a=0$

$(\mathbb{Z},\cdot)$ is not a group because for

④ we need $b=\frac{1}{a}$ but $\frac{1}{a} \notin \mathbb{Z}$

$(\mathbb{R},\cdot)$ is not a group because for

④ when $a=0$ then there is no b with

$0\cdot b=b\cdot 0=1$

$(\mathbb{R}^*,\cdot)$ where $\mathbb{R}^*=\mathbb{R}\setminus\{0\}$ is a group

since we know $a \neq 0$ so $a\cdot\frac{1}{a}=1$

The following are groups

- $\mathbb{Q}=\{\frac{a}{b} \mid a,b \in \mathbb{Z}, b \neq 0\}$
- $(\mathbb{Q}\setminus\{0\},\cdot)$ is a group
- $(\mathbb{Q},+)$ is a group
- $(\mathbb{C},+)$ is a group

Wednesday week 2 aug. 31, 2016

Example: $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1}\}$ is a group under +

Let $\bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}_n$ where $a, b, c \in \mathbb{Z}$

(1) Then $\bar{a} + \bar{b} = \overline{a+b}$ is in $\mathbb{Z}_n$ b/c $a+b \in \mathbb{Z}$

(2) also $\bar{a} + (\bar{b} + \bar{c}) = \bar{a} + (\overline{b+c}) = \overline{a+(b+c)} = \overline{(a+b)+c}$

$= (\overline{a+b}) + \bar{c} = (\bar{a} + \bar{b}) + \bar{c}$

(3) $\bar{a} + \bar{0} = \overline{a+0} = \bar{a}$ and $\bar{0} + \bar{a} = \overline{0+a} = \bar{a}$

(4) we have $\bar{a} + -\bar{a} = \overline{a + (-a)} = \bar{0}$

$-\bar{a} + \bar{a} = \overline{-a+a} = \bar{0}$

so $\bar{a}$ has an inverse under addition.

Example: $U = \{z \in \mathbb{C} \mid |z| = 1\}$

the unit circle in $\mathbb{C}$ is a group under multiplication

Example $U_n = \{z \in \mathbb{C} \mid z^n = 1\}$

$= \{\xi^k \mid k = 0, 1, \ldots, n-1\}$

where $\xi = e^{2\pi/n\, i}$

U_n is a group under multiplication

Proof

(1) Let $z, w \in U_n$ Then $z^n = 1$ and $w^n = 1$

so $(zw)^n = z^n w^n = 1 \cdot 1 = 1$

so $z \cdot w \in U_n$

(2) Let $z, w, y \in U_n$, since z, w, y are complex numbers we know $(z \cdot w) \cdot y = z(w \cdot y)$

(3) $1 \in U_n$ since $1^n = 1$

(4) Let $z \in U_n$ then $z^n = 1$

so $1 = \frac{1}{z^n} = \left(\frac{1}{z}\right)^n$ so $\frac{1}{z} \in U_n$ * note

$z \neq 0$ since $0^n \neq 1$

and $z \cdot \frac{1}{z} = 1$ likewise $\frac{1}{z} \cdot z = 1$

Abelian

Let $(G, *)$ be a group, we say G is abelian if $a * b = b * a \quad \forall\, a, b \in G$

all the groups we have seen so far are abelian.

Example

$$GL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} a, b, c, d \in \mathbb{R} \\ ad - bc \neq 0 \end{array} \right\} \leftarrow \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$$

$GL(2, \mathbb{R})$ is a group under multiplication

(1) Recall that $\underbrace{\det(AB)}_{\neq 0} = \underbrace{\det(A)}_{\neq 0} \underbrace{\det B}_{\neq 0}$

(2) ✓

(3) $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(4) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \leftarrow$ exists when $ad - bc \neq 0$