<u>ex:</u> $\langle U, \cdot \rangle$ is a group.

<u>ex:</u> $\langle U_n, \cdot \rangle$ is a group.

<u>ex:</u> $\langle U_n, + \rangle$ is not a group.

<u>ex:</u> $\langle \mathbb{Z} \setminus \{0\}, \cdot \rangle$ is not a group

<u>ex:</u> $\langle \mathbb{Q}, + \rangle$

<u>ex:</u> $\langle \mathbb{Q} \setminus \{0\}, \cdot \rangle$

<u>Def:</u> A group G is <u>abelian</u> if $a * b = b * a$ for all $a, b \in G$.

<u>ex:</u> $\langle \mathbb{Z}, + \rangle$ is abelian

<u>ex:</u> $\langle GL(2,\mathbb{R}), \cdot \rangle$ is not abelian.

$$GL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d \in \mathbb{R} \\ \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \end{matrix} \right\}$$

Use $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ & $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

<u>Lemma: (Cancellation ~~law~~ in groups).</u>

Let $\langle G, * \rangle$ be a group.

(i) If $a * b = c * b$, then $a = c$.

(ii) If $b * a = b * c$, then $a = c$.

<u>pf:</u> (i) Suppose $a*b = c*b$. Since G is a group, there exists $b^{-1} \in G$ such that $b^{-1} * b = b * b^{-1} = e$.

Thus,

$$\begin{aligned}
(a*b) &= (c*b) \\
(a*b)*b^{-1} &= (c*b)*b^{-1} \\
a*(b*b^{-1}) &= c*(b*b^{-1}) \\
a*e &= c*e \Rightarrow a = c.
\end{aligned}$$

associativity in G

(ii) Same. $\blacksquare$

9/7 P.1 Wednesday Week 3 Sept. 7, 2016

missing Mon. W3 sep. 5. 16

Review (HW #1 Prob. #7)

⑦ $G = \mathbb{R} \setminus \{-1\}$

$a * b = a + b + ab$

Prove G is a group under $*$

Proof

① (closure) Suppose $x, y \in G = \mathbb{R}\setminus\{-1\}$ so $x, y \in \mathbb{R} \setminus \{-1\}$

that is $x \neq -1$ and $y \neq -1$

we need to show that $x * y = x + y + xy$ is also $\in \mathbb{R}\setminus\{-1\}$

Suppose (by contradiction) that $x + y + xy = -1$

But then solving for x would give

$x = \dfrac{-1-y}{1+y} = -1$ which contradicts $x \neq -1$

$1+y$ not zero since $y \neq -1$

② (associativity)

Let $x, y, z \in \mathbb{R}\setminus\{-1\}$

then $x * (y * z) = x + y*z + x(y*z)$

$= x + (y + z + yz) + x(y + z + yz)$

and $(x * y) * z = x * y + z + (x*y)z$

$= (x + y + xy) + z + (x + y + xy)z$

these 2 are equal

③ (identity)

Let $x \in \mathbb{R}\setminus\{-1\}$

then $x * 0 = x + 0 + x \cdot 0 = x$

$0 * x = 0 + x + 0 \cdot x = x$

so 0 is an identity element for $\mathbb{R}\setminus\{-1\}$

scratch work

$x * e = x + e + xe$

$x = \dfrac{0}{1+x} = 0$

④ (inverses)

Let $x \in \mathbb{R}\setminus\{-1\}$

Lets show that $\frac{-x}{1+x}$ is an inverse for x.

we have that

$x * \left(\frac{-x}{1+x}\right) = x + \left(\frac{-x}{1+x}\right) + x\left(\frac{-x}{1+x}\right) = \dfrac{(x+x^2)-x-x^2}{1+x} = 0$

and $\left(\frac{-x}{1+x}\right) * x = \left(\frac{-x}{1+x}\right) + x + \left(\frac{-x}{1+x}\right)x = 0$ ☒

scratch work

$x * y = 0$ e

$x + y + xy = 0$

$y = \dfrac{-x}{1+x}$

Continuation of last time...

Proposition: Let $(G, *)$ be a group

Then

(1) There is only one identity element in G. (e is unique)

(2) For each $a \in G$ $\exists$ a <u>unique</u> $b \in G$ where

$a * b = b * a = e$.

• we will write a^{-1} for this unique b.

$\cancel{a}$ a^{-1}

(3) For each $a \in G$, we have $(a^{-1})^{-1} = a$

(4) For each $a, b \in G$ we have $(a * b)^{-1} = (b^{-1}) * (a^{-1})$

Proof

① Suppose G has 2 identity elements e and f that both satisfy axiom 3 of a group.

Then $e = e * f = f$

since e satisfies axiom 3

since f satisfies axiom 3

thus $e = f$

② Let $a \in G$. Suppose b and c are both inverses for a. That is, $b * a = a * b = e$ and $c * a = a * c = e$,

then $b * a = e$

so $(b * a) * c = e * c$

thus $b * (\underbrace{a * c}_{e}) = c$

$\underbrace{b * e}_{b} = c \implies b = c$ thus a has a unique inverse in G.

③ $(a^{-1})^{-1} = a$ since

$a * a^{-1} = a^{-1} * a = e$ } This says that a is the inverse of a^{-1}

Sidenote

x^{-1} means $x^{-1} * x = x * x^{-1} = e$

to show x and y are inverses we show

$x * y = y * x = e$

9/7 P. 2

④ Let $a, b \in G$

star it with its inverse

Then $[a * b] * \underbrace{[b^{-1} * a^{-1}]} \overset{\text{by associativity}}{=} a * \underbrace{b * b^{-1}} * a^{-1}$

$a * e * a^{-1}$

$a * a^{-1} = e$

and $[b^{-1} * a^{-1}] * [a * b] \overset{\text{By Associativity}}{=} b^{-1} * \underbrace{a^{-1} * a} * b$

$= b^{-1} * e * b$

$= b^{-1} * b = e$

so $(a * b)^{-1} = b^{-1} * a^{-1}$ ☒

Notation

- when dealing with a group $(G, *)$. we will write <u>ab</u> instead of $a * b$
  does not necessarily mean multiplication it depends on $*$
- also we will just write, "Let G be a group" without mentioning $*$
- we don't need parenthesis by associativity so we write stuff like $abacc$
  $= a * b * a * c * c$

Let G be a group with identity e and let x be in G.

define

- $x^0 = e$
- If n is a positive integer then $x^n = \underbrace{xxx\cdots x}_{n \text{ times}}$ and $x^{-n} = \underbrace{x^{-1}x^{-1}x^{-1}\cdots x^{-1}}_{n \text{ times}}$

Example

$x^3 = xxx = x * x * x$

$x^{-4} = x^{-1}x^{-1}x^{-1}x^{-1}$

$x^0 = e$

$a^3b^{-2} = aaab^{-1}b^{-1}$