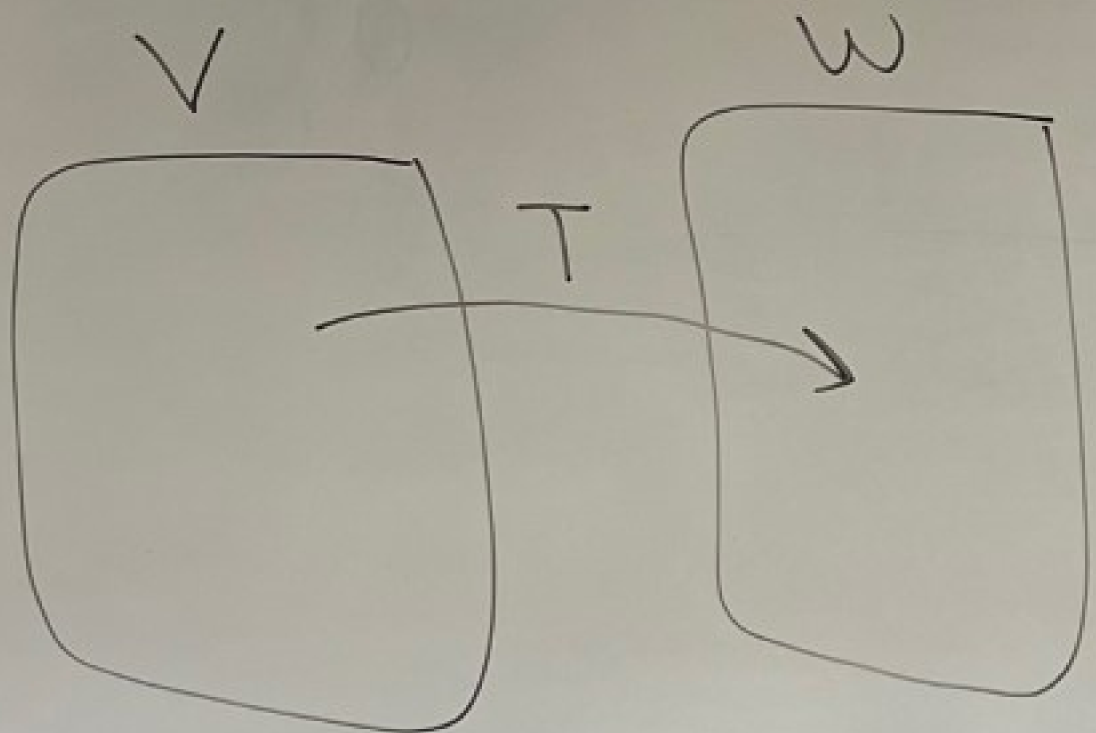
Def: Let V and W be vector spaces over a field F.

① An isomorphism between V and W is a linear transformation $T: V \to W$ that is one-to-one and onto.



② We say that V and W are isomorphic if there exists an isomorphism $T: V \to W$. If V and W are isomorphic then we write $V \cong W$

Ex: $V = \mathbb{R}^2$, $W = P_1(\mathbb{R})$, $F = \mathbb{R}$

Let $T : \mathbb{R}^2 \to P_1(\mathbb{R})$ be given by $T(a,b) = a + bx$

$$\mathbb{R}^2 \xrightarrow{T} P_1(\mathbb{R})$$

$$\begin{aligned}
(a,b) &\longmapsto a+bx \\
(1,3) &\longmapsto 1+3x \\
(-2,-1) &\longmapsto -2-x \\
(1,3)+(-2,-1)=(-1,2) &\longmapsto -1+2x = (1+3x)+(-2-x) \\
7\cdot(1,3)=(7,21) &\longmapsto 7+21x = 7(1+3x)
\end{aligned}$$

$\boxed{\text{T is an isomorphism}}$

$$\left.\begin{array}{l} T \text{ is one-to-one} \\ T \text{ is onto} \\ T \text{ is linear} \end{array}\right\} \text{you can check these}$$

T shows that $\mathbb{R}^2$ and $P_1(\mathbb{R})$ are the same structurally / algebraically
it's just that the names of the objects are different on each side.

$$\boxed{\text{So, } \mathbb{R}^2 \cong P_1(\mathbb{R}).}$$

Theorem: Let V, W, Z be vector spaces over a field F. Then:

① If $V \cong W$, then $W \cong V$.

② If $V \cong W$ and $W \cong Z$, then $V \cong Z$.

proof sketch (needs details):

① $V \cong W \Rightarrow \exists\, T: V \to W$ that is an isomorphism

$\Rightarrow T^{-1}: W \to V$ is an isomorphism $\Rightarrow W \cong V$.

② $V \cong W$ and $W \cong Z \Rightarrow \exists\, T_1: V \to W,\ T_2: W \to Z$ that are both isomorphisms

$\Rightarrow T_2 \circ T_1: V \to Z$ is an isomorphism $\Rightarrow V \cong Z$ ▨

Theorem (Constructing linear transformations)

Let V and W be vector spaces over a field F.
Suppose V is finite dimesional with basis $\beta = \{v_1, v_2, \dots, v_n\}$.

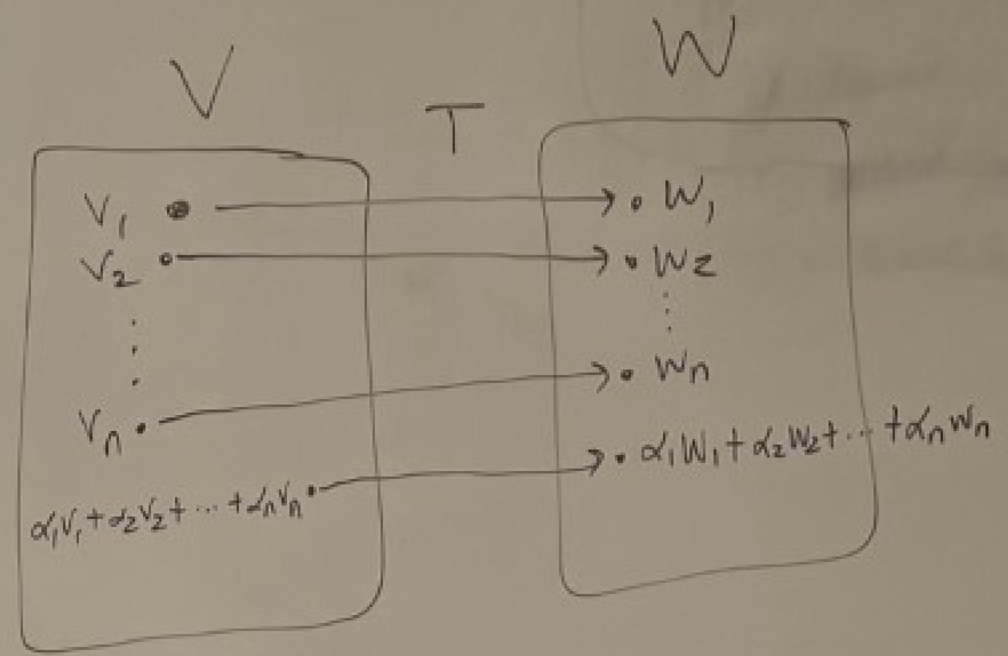
PART 1

Pick any $w_1, w_2, \dots, w_n \in W$.

① Then there exists a unique linear transformation $T: V \to W$ with $T(v_1) = w_1$, $T(v_2) = w_2$, $\dots$, $T(v_n) = w_n$ given by the formula

$$T(\underbrace{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n}_{\substack{\text{any element of } V \\ \text{looks like this} \\ \text{since } \beta \text{ is a basis for } V}}) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n \qquad (*)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n \in F$



(2) T given above in $(*)$ is an isomorphism if and only if $\beta' = \{w_1, w_2, \dots, w_n\}$ is a basis for W

$\boxed{\text{PART 2}}$ All linear transformations $T: V \to W$ are constructed as in part 1 #1.

That is, if $T: V \to W$ is a linear transformation then set $w_1 = T(v_1)$, $w_2 = T(v_2)$, ..., $w_n = T(v_n)$ and the formula for T will be $(*)$, that is

$$T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n$$

<u>Proof:</u> I'll post online.

Ex: Let's build a linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^2$

Here $V=\mathbb{R}^3$, $W=\mathbb{R}^2$, $F=\mathbb{R}$.

Pick a basis for $\mathbb{R}^3$: $\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{v_3}$

Pick where this basis goes.

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix} \leftarrow w_1$$

$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix} \leftarrow w_2$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix} \leftarrow w_3$$

$\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^2$

$$v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \longrightarrow \begin{pmatrix}1\\-3\end{pmatrix} = w_1$$

$$v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \longrightarrow \begin{pmatrix}0\\2\end{pmatrix} = w_2$$

$$v_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} \longrightarrow \begin{pmatrix}1\\4\end{pmatrix} = w_3$$

$$\begin{pmatrix}a\\b\\c\end{pmatrix} \longrightarrow \begin{pmatrix}a+c\\-3a+2b+4c\end{pmatrix}$$

Get a formula for T

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = T\left(a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$= aT\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + bT\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + cT\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

need T to be linear

$$= a\begin{pmatrix} 1 \\ -3 \end{pmatrix} + b\begin{pmatrix} 0 \\ 2 \end{pmatrix} + c\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} a+c \\ -3a+2b+4c \end{pmatrix}$$

So, $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+c \\ -3a+2b+4c \end{pmatrix}$ is the unique linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ with $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

By the theorem, T is an isomorphism iff $w_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $w_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $w_3 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ are a basis for $\mathbb{R}^2$.

We have three vectors w_1, w_2, w_3 in a 2-dimensional vector space $\mathbb{R}^2$.

By a previous thm w_1, w_2, w_3 must be lin. dep. and thus not a basis.

So, T is <u>not</u> an isomorphism.

So, $\mathbb{R}^3$ is <u>not</u> isomorphic to $\mathbb{R}^2$.

$(F=\mathbb{R})$

Ex: Let $T:\mathbb{R}^2 \longrightarrow P_1(\mathbb{R})$

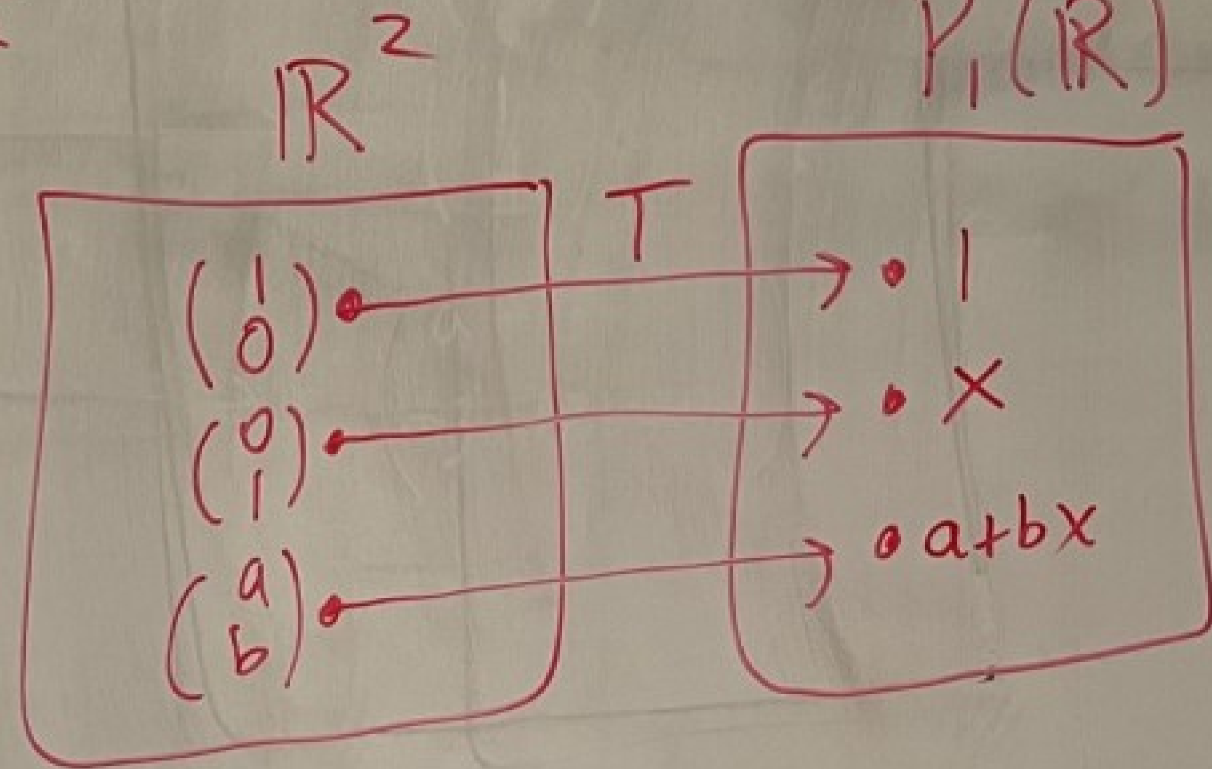
be constructed as follows:

pick basis for $\mathbb{R}^2$: $\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{v_2}$

pick where the basis goes:

$T\begin{pmatrix}1\\0\end{pmatrix} = 1 \leftarrow w_1$

$T\begin{pmatrix}0\\1\end{pmatrix} = x \leftarrow w_2$



formula for T

$$T\begin{pmatrix} a \\ b \end{pmatrix} = T\left(a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= aT\begin{pmatrix} 1 \\ 0 \end{pmatrix} + bT\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= a \cdot 1 + b \cdot x$$

$$= a + bx$$

So, $\boxed{T\begin{pmatrix} a \\ b \end{pmatrix} = a + bx}$

T is an isomorphism
iff $w_1 = 1, w_2 = x$
is a basis for $P_1(\mathbb{R})$.

$w_1 = 1, w_2 = x$ is
the standard basis
for $P_1(\mathbb{R})$.

So, T is an isomorphism
and so $\mathbb{R}^2 \cong P_1(\mathbb{R})$.

Theorem: Let V and W be finite-dimensional vector spaces over a field F.

Then: $V \cong W$ iff $\dim(V) = \dim(W)$.

proof:

($\Rightarrow$) Suppose $V \cong W$.

Then there exists $T: V \to W$ where T is a one-to-one and onto linear transformation.

Since T is 1-1, by HW this means $N(T) = \{\vec{0}_V\}$ and $\dim(N(T)) = 0$.

Since T is onto, we know $W = R(T)$. So, $\dim(R(T)) = \dim(W)$.

By the rank-nullity theorem

$$\dim(V) = \dim(N(T)) + \dim(R(T)) = 0 + \dim(W) = \dim(W).$$

($\Leftarrow$) next time... $\blacksquare$