Hw 2 – 7(b)
Hw 2 – 9] NOT ON TEST

Hw 2 #1

$V = M_{2,2}(\mathbb{R}) \quad F = \mathbb{R}$

$W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \,\middle|\, a,b,c \in \mathbb{R} \right\}$

$W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \,\middle|\, a,b \in \mathbb{R} \right\}$

(a) Show W_1, W_2 are subspaces ✓

(b) Find dimensions of $W_1, W_2, W_1 \cap W_2$, $\underline{\underline{W_1 + W_2}}$

$\dim(W_1) = ?$

Let $\begin{pmatrix} a & b \\ c & a \end{pmatrix} \in W_1$

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

So every vector in W_1 is in the span of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Then show $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are linearly independent.

Once you've done that a basis for W_1 is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

So $\dim(W_1) = 3$

$\dim(W_2) = 2$ (See sols)

$\dim(W_1 \cap W_2) = 1$ (see sols)

$$\boxed{\begin{array}{l} W_1 + W_2 = \{x + y \mid x \in W_1, y \in W_2\} \\ \text{What is } \dim(W_1 + W_2). \; ? \end{array}}$$

Let $w \in W_1 + W_2$.

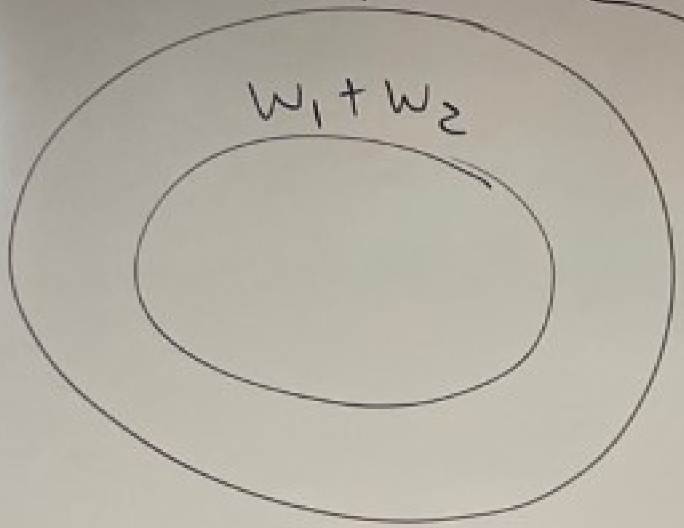
Then $w = x + y$ where $x \in W_1$, $y \in W_2$.

So, $w = \underbrace{\begin{pmatrix} a & b \\ c & a \end{pmatrix}}_{\text{in } W_1} + \underbrace{\begin{pmatrix} 0 & \alpha \\ -\alpha & \beta \end{pmatrix}}_{\text{in } W_2}$ where $a, b, c, \alpha, \beta \in \mathbb{R}$

$$= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \beta\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

HW 2 – 7(b)]
HW 2 – 9] NOT ON TEST

$M_{2,2}(\mathbb{R}) \leftarrow \boxed{\text{dim } 4}$



So, $\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \overbrace{\begin{pmatrix}0&1\\-1&0\end{pmatrix}}^{\text{can eliminate}}, \begin{pmatrix}0&0\\0&1\end{pmatrix}$

span $W_1 + W_2$.

Since $\dim(M_{2,2}(\mathbb{R})) = 4$, these 5 vectors are lin. dep. and can't be a basis for $W_1 + W_2$.

Note: $\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix} - \begin{pmatrix}0&0\\1&0\end{pmatrix}$

$M_{m,n}(\mathbb{R})$

Thus,

$W_1 + W_2 = \text{span}\left(\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}\right) = \text{span}\left(\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}\right)$

since $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ is in here

You can show that

$\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}$

are lin. ind. Why?

Suppose

$$c_1\begin{pmatrix}1&0\\0&1\end{pmatrix} + c_2\begin{pmatrix}0&1\\0&0\end{pmatrix} + c_3\begin{pmatrix}0&0\\1&0\end{pmatrix} + c_4\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

This becomes

$$\begin{pmatrix}c_1&0\\0&c_1\end{pmatrix} + \begin{pmatrix}0&c_2\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\c_3&0\end{pmatrix} + \begin{pmatrix}0&0\\0&c_4\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

This gives

$$\begin{pmatrix}c_1&c_2\\c_3&c_1+c_4\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

So,

$$\boxed{\begin{aligned} c_1 \qquad &= 0 \quad ① \\ c_2 \qquad &= 0 \quad ② \\ c_3 \qquad &= 0 \quad ③ \\ c_1 \qquad + c_4 &= 0 \quad ④ \end{aligned}}$$

①, ②, ③ force $c_1=0, c_2=0, c_3=0$.

So ④ becomes $\underbrace{0}_{c_1} + c_4 = 0 \rightarrow c_4 = 0$.

Thus, $c_1=c_2=c_3=c_4=0$ is the only sol.

So, $\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}$ is a basis for W_1+W_2.

So, $\dim(W_1+W_2)=4$.

Since $\dim(M_{2,2}(\mathbb{R}))=4$ and $\dim(W_1+W_2)=4$ and W_1+W_2 is a subspace of $M_{2,2}(\mathbb{R})$ they must be equal.

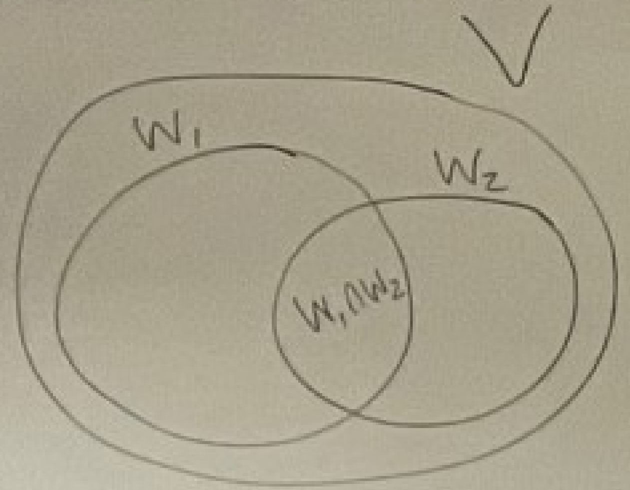
So, $W_1+W_2 = M_{2,2}(\mathbb{R})$.

HW 2 #8

Suppose V is a vector space over a field F.
Suppose W_1 is a subspace of V with $\dim(W_1) = m$
Suppose W_2 is a subspace of V with $\dim(W_2) = n$

(a) Show $\dim(W_1 \cap W_2) \leq \min\{\dim(W_1), \dim(W_2)\}$
(b) Show $\dim(W_1 + W_2) \leq m + n$

(a)



Since $W_1 \cap W_2$ is a subspace of W_1
we know $\dim(W_1 \cap W_2) \leq \dim(W_1)$

Since $W_1 \cap W_2$ is a subspace of W_2
we know $\dim(W_1 \cap W_2) \leq \dim(W_2)$

So, $\dim(W_1 \cap W_2) \leq \min\{\dim(W_1), \dim(W_2)\}$

(b) If $W_1 = \{\vec{0}\}$, then $W_1 + W_2 = W_2$ and so $\dim(W_1 + W_2) = \dim(W_2) = n \leq m + n$
If $W_2 = \{\vec{0}\}$, then $W_1 + W_2 = W_1$ and so $\dim(W_1 + W_2) = \dim(W_1) = m \leq n + m$
So assume $W_1 \neq \{\vec{0}\}$ and $W_2 \neq \{\vec{0}\}$
Since W_1 has dimension m, there exists a basis
$v_1, v_2, \dots, v_m \in W_1$ for W_1.
Since W_2 has dimension n, there exists a basis
$w_1, w_2, \dots, w_n \in W_2$ for W_2.

Pick $x \in W_1 + W_2$.
Then $x = a + b$ where $a \in W_1$ and $b \in W_2$.
So, $x = \underbrace{c_1 v_1 + c_2 v_2 + \cdots + c_m v_m}_{a} + \underbrace{d_1 w_1 + d_2 w_2 + \cdots + d_n w_n}_{b}$

So, $v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_n$ span $W_1 + W_2$.

From HW 2 #7(b) there must exist a basis

β for $W_1 + W_2$ where $\beta \subseteq \{v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_n\}$

So, $|\beta| \leq m+n$.

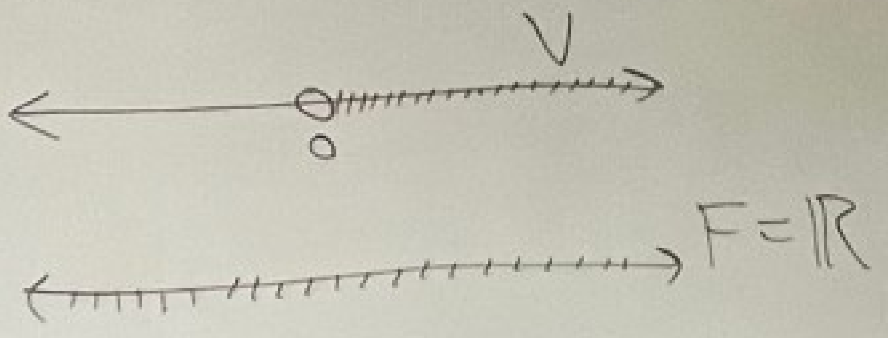
Thus, $\dim(W_1 + W_2) \leq m+n$. $\blacksquare$

HW 1

① (c) Is V a vector space over F?

$V = \{x \mid x \in \mathbb{R}, x > 0\}$

$F = \mathbb{R}$



Define vector adding:

Given $x, y \in V$ define $x \oplus y = xy$

Given $\alpha \in F$, $x \in V$ define $\alpha \odot x = x^{\alpha}$

Ex:

$2 \oplus 5 = (2)(5) = 10$

$2 \odot 5 = 5^2 = 25$

This is a vector space.

You'd have to verify (V1) – (V8)

(V8) $(a+b)v = av + bv$

for example

(V8) Let $\alpha, \beta \in F$, $v \in V$.

Then,

$$\begin{aligned}(\alpha+\beta) \odot v &= v^{\alpha+\beta} \\ &= v^{\alpha} v^{\beta} \\ &= v^{\alpha} \oplus v^{\beta} \\ &= (\alpha \odot v) \oplus (\beta \odot v)\end{aligned}$$