<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F.

Let $T: V \to V$ be a linear transformation.

Let β be an ordered basis for V.

Then,

$$\det(T - \lambda I) = \det([T]_\beta - \lambda I_n)$$

where I_n is the $n \times n$ identity matrix

and $n = \dim(V)$.

$$\boxed{\begin{aligned} &I: V \to V \\ &I(x) = x \quad \forall x \in V \end{aligned}}$$

$$\boxed{\begin{aligned} (T-\lambda I)(x) &= T(x) - \lambda I(x) \\ &= T(x) - \lambda x \\ T-\lambda I &: V \to V \end{aligned}}$$

proof: We have that

$$\det(T-\lambda I) = \det([T-\lambda I]_\beta) = \det([T]_\beta - \lambda [I]_\beta) = \det([T]_\beta - \lambda I_n)$$

def of det

Hw 4 #2

$[S+T]_\beta = [S]_\beta + [T]_\beta$

$[cT]_\beta = c[T]_\beta$

Hw 5 #2

$[I]_\beta = I_n$

$\blacksquare$

Def: Let V be a finite-dimensional vector space over a field F.

Let $T: V \to V$ be a linear transformation.

Let λ be an eigenvalue of T.

Define

$$E_\lambda(T) = \{x \in V \mid T(x) = \lambda x\}$$

$$= N(T - \lambda I)$$

$$\left\{\begin{aligned} T(x) &= \lambda x \\ T(x) - \lambda x &= \vec{0} \\ T(x) - \lambda I(x) &= \vec{0} \\ (T - \lambda I)(x) &= \vec{0} \end{aligned}\right.$$

$E_\lambda(T)$ is called the <u>eigenspace of T corresponding to λ</u>.

The dimension of $E_\lambda(T)$ is called the <u>geometric multiplicity of λ</u>.

Note:
- $E_\lambda(T)$ is a subspace of V [HW 5]
- $E_\lambda(T)$ consists of $\vec{0}$ and all the eigenvectors for λ.

__Def:__ Let V be a finite-dimensional vector space over a field F.
Let $T: V \to V$ be a linear transformation.
Let β be an ordered basis for V. Let $n = \dim(V)$.
Then the function

$$f_T(\lambda) = \det(T - \lambda I) = \det([T]_\beta - \lambda I_n)$$

is called the <u>characteristic polynomial of T.</u> The roots of $f_T(\lambda)$ are the eigenvalues of T. If λ_0 is a root of $f_T(\lambda)$ then it's multiplicity as a root is called the <u>algebraic multiplicity of λ_0</u>

That is, the algebraic multiplicity of λ_0 is the largest positive integer k where $(\lambda - \lambda_0)^k$ is a factor of $f_T(\lambda)$

<u>Ex</u>: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix}$

You can show this is a linear transformation.

Let's find all the eigenvalues, eigenvectors, etc for T.

Let $\beta = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$

Let's find $[T]_\beta = [T]_\beta^\beta$

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Longrightarrow \quad [T]_\beta = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

So,

$$f_T(\lambda)=\det(T-\lambda I)=\det([T]_\beta-\lambda I_3)=\det\left(\begin{pmatrix}0&0&-2\\1&2&1\\1&0&3\end{pmatrix}-\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)$$

$$=\det\begin{pmatrix}-\lambda&0&-2\\1&2-\lambda&1\\1&0&3-\lambda\end{pmatrix}=-0+(2-\lambda)\begin{vmatrix}-\lambda&-2\\1&3-\lambda\end{vmatrix}-0=(2-\lambda)\left[(-\lambda)(3-\lambda)-(-2)(1)\right]$$

$$\begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix} \qquad \begin{pmatrix}-\lambda&0&-2\\1&2-\lambda&1\\1&0&3-\lambda\end{pmatrix}$$

$$=-\lambda^3+5\lambda^2-8\lambda+4$$

<u>Rational roots theorem</u> Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ where $a_n, a_{n-1}, \dots, a_1, a_0$ are integers, and $a_n \neq 0$, $a_0 \neq 0$. If $\frac{p}{q}$ is a root of $f(x)$, then p divides a_0 and q divides a_n.

The possible rational roots of $f_T(\lambda) = -\lambda^3 + 5\lambda - 8\lambda + 4$ are

-1 $\quad$ 4

$\frac{p}{q}$ where p divides 4 and q divides -1.

So, $p = \pm 1, \pm 2, \pm 4$ and $q = \pm 1$.

So, $\frac{p}{q} = \pm 1, \pm 2, \pm 4$.

Try them out one by one till you find a root (hopefully)

$$f_T(1) = -1 + 5 - 8 + 4 = 0$$
$$f_T(-1) = 16 \neq 0$$
$$f_T(2) = 0$$
$$f_T(-2) \neq 0$$
$$f_T(\pm 4) \neq 0$$

Thus, the only rational roots are 1 and 2

Since $f_T(1)=0$ we know $(\lambda-1)$ divides $f_T(\lambda)$.

$$
\begin{array}{r}
-\lambda^2+4\lambda-4 \\
\lambda-1 \,\big)\overline{-\lambda^3+5\lambda^2-8\lambda+4} \\
-(-\lambda^3+\lambda^2) \\
\hline
4\lambda^2-8\lambda+4 \\
-(4\lambda^2-4\lambda) \\
\hline
-4\lambda+4 \\
-(-4\lambda+4) \\
\hline
0
\end{array}
$$

So,

$$
\begin{aligned}
f_T(\lambda) &= -\lambda^3+5\lambda^2-8\lambda+4 \\
&= (\lambda-1)(-\lambda^2+4\lambda-4) \\
&= -(\lambda-1)(\lambda^2-4\lambda+4) \\
&= -(\lambda-1)(\lambda-2)(\lambda-2) \\
&= -(\lambda-1)^1(\lambda-2)^2
\end{aligned}
$$

Summary so far

Eigenvalue λ	algebraic multiplicity of λ
$\lambda=1$	1
$\lambda=2$	2

$\boxed{\text{Let's find a basis for } E_1(T)} \leftarrow \boxed{\lambda = 1}$

$\boxed{E_\lambda(T) = \{x \mid T(x) = \lambda x\}}$

$$E_1(T) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; \begin{pmatrix} -a-2c \\ a+b+c \\ a+2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Need to solve $\boxed{\begin{aligned} -a \quad\;\; -2c &= 0 \\ a + b + c &= 0 \\ a \quad\;\; +2c &= 0 \end{aligned}}$

Let's solve

make into 1

make into 0's

$$\left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \xrightarrow{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ \underbrace{0 \quad 0 \quad 0}_{\text{reduced } \checkmark} & & & 0 \end{array}\right)$$

Convert back to system:

$$\boxed{\begin{aligned} a \quad\quad +2c &= 0 \quad ① \\ b - c &= 0 \quad ② \\ 0 &= 0 \quad \checkmark \end{aligned}}$$

leading variables: a, b
free variable: c

⇒

solve for leading
give free var. new name

$$\boxed{\begin{aligned} a &= -2c \quad ① \\ b &= c \quad ② \\ c &= t \quad ③ \end{aligned}}$$

⇒

back-substitute

$$\boxed{\begin{aligned} &③\ c = t \\ &②\ b = c = t \\ &①\ a = -2c = -2t \end{aligned}}$$

⇒

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in E_1(T)$$

iff

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

⇓

basis for $E_1(T)$ is $\beta_1 = \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right]$

$\dim(E_1(T)) = 1 =$ geometric multiplicity of $\lambda = 1$