Math 4570

12/5/22

Last thing about eigenvalues

Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation.

Then:

① Let λ be an eigenvalue of T. Then,

$$1 \le \underbrace{\begin{pmatrix}\text{geometric multiplicity} \\ \text{of } \lambda\end{pmatrix}}_{\dim(E_\lambda(T))} \le \underbrace{\begin{pmatrix}\text{algebraic multiplicity} \\ \text{of } \lambda\end{pmatrix}}_{\text{multiplicity of } \lambda \text{ as a root of the characteristic polynomial}}$$

② T is diagonalizable iff

$$\begin{pmatrix}\text{geometric mult.} \\ \text{of } \lambda\end{pmatrix} = \begin{pmatrix}\text{alg. mult} \\ \text{of } \lambda\end{pmatrix}$$

for all eigenvalues λ of T

HW 5

① (c) $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a+b \\ 3b \\ 4c \end{pmatrix}$

(i) Find the eigenvalues of T

Let $\beta = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$.

Let's calculate $[T]_\beta$.

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So,

$$[T]_\beta = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Thus,

$$f_T(\lambda) = \det\left([T]_\beta - \lambda I_3\right)$$

$$= \det\left(\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

expand on column 1

$$= (3-\lambda) \begin{vmatrix} 3-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} - 0 \begin{vmatrix} 1 & 0 \\ 0 & 4-\lambda \end{vmatrix} + 0 \begin{vmatrix} 1 & 0 \\ 3-\lambda & 0 \end{vmatrix}$$

$$\begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix} \quad \begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix} \quad \begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix}$$

$$= (3-\lambda)\left[(3-\lambda)(4-\lambda) - (0)(0)\right] + 0 + 0$$

$$= \boxed{(3-\lambda)(3-\lambda)(4-\lambda)}$$

$$= \boxed{-(\lambda-3)^2(\lambda-4)}$$

Eigenvalue	algebraic multiplicity
$\lambda = 3$	2
$\lambda = 4$	1

(ii) Find a basis for each eigenspace.

Let's start with $E_3(T)$

We have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in E_3(T) \quad \text{iff} \quad T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 3\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\text{iff} \quad \begin{pmatrix} 3a+b \\ 3b \\ 4c \end{pmatrix} = \begin{pmatrix} 3a \\ 3b \\ 3c \end{pmatrix}$$

$$\text{iff} \quad \begin{pmatrix} b \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $\boxed{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in E_3(T)}$ iff $\boxed{\begin{matrix} b=0 \\ c=0 \\ a \in \mathbb{R} \end{matrix}}$

So, $\boxed{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in E_3(T)}$

iff $\boxed{\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

So, $E_3(T) = \text{span}\left(\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}\right)$.

Let $\beta_1 = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]$ ← one non-zero vector Hw 2 #6, this is a basis

What about $E_4(T)$?

$$E_4(T) = \left\{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 4\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \begin{pmatrix} 3a+b \\ 3b \\ 4c \end{pmatrix} = \begin{pmatrix} 4a \\ 4b \\ 4c \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \begin{pmatrix} -a+b \\ -b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \begin{matrix} -a+b=0 \\ -b=0 \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \underbrace{\begin{matrix} a-b=0 \\ b=0 \end{matrix}}_{\text{reduced system}} \right\}$$

$$\boxed{\begin{matrix} a-b=0 \\ b=0 \end{matrix}}$$

leading variables: a, b

free variables: c

$$\boxed{\begin{matrix} a=b \\ b=0 \\ c=t \end{matrix}} \quad \boxed{t \in \mathbb{R}}$$

So, $\boxed{\begin{aligned} &c = t,\ t \in \mathbb{R}. \\ &b = 0 \\ &a = b = 0. \end{aligned}}$

Thus,

$$E_4(T) = \left\{ \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} \middle| \ t \in \mathbb{R} \right\}$$

$$= \left\{ t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| \ t \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left(\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \right)$$

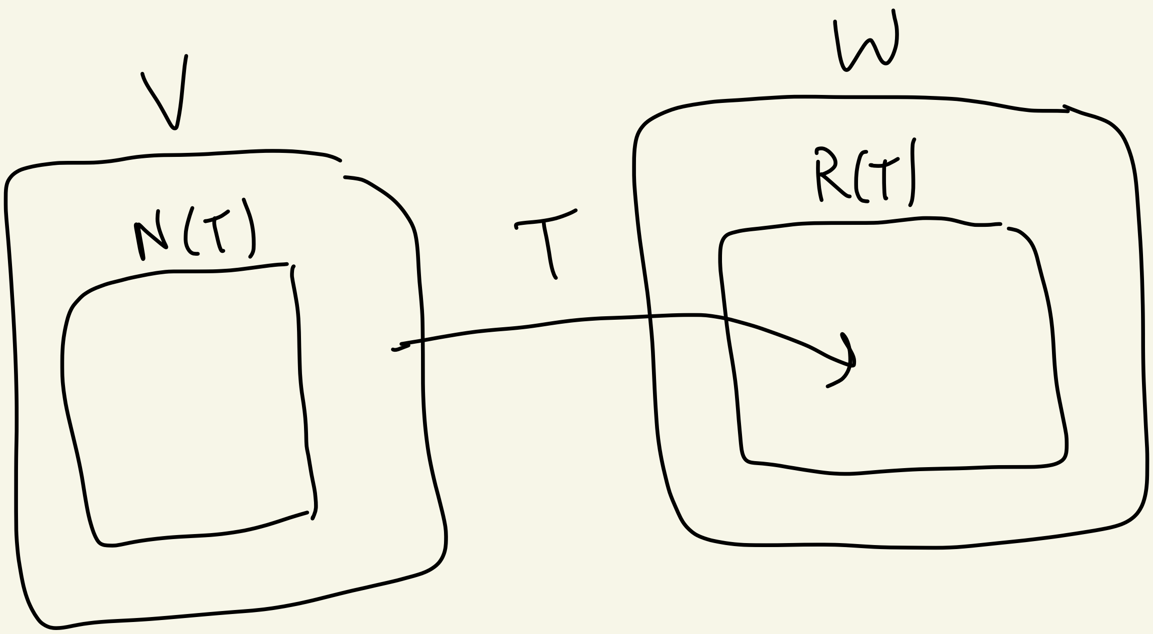
basis for $E_4(T)$ is $\beta_2 = \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$

Summary

Eigenvalue	basis for $E_\lambda(T)$	geometric mult.	alg. mult.
3	$\beta_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$	1	2
4	$\beta_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$	1	1

Note: T is not diagonalizable because (geometric mult of $\lambda = 3$) $\neq$ (algebraic mult of $\lambda = 3$)

$T: V \to W$ lin. transformation



One-to-one: T is 1-1 iff $N(T) = \{\vec{0}\}$

iff $\underbrace{\dim(N(T))}_{\text{nullity of } T} = 0$

onto: T is onto iff $R(T) = W$

iff $\underbrace{\dim(R(T))}_{\text{rank}(T)} = \dim(W)$

rank-nullity thm

$$\dim(V) = \dim(N(T)) + \dim(R(T))$$