Theorem: Let V be a vector space over a field F. Suppose $\dim(V) = n > 0$. Then the following are true:

① Let $v_1, v_2, \ldots, v_m \in V$.

(a) If $m > n$, then $v_1, v_2, \ldots, v_m$ are linearly dependent.

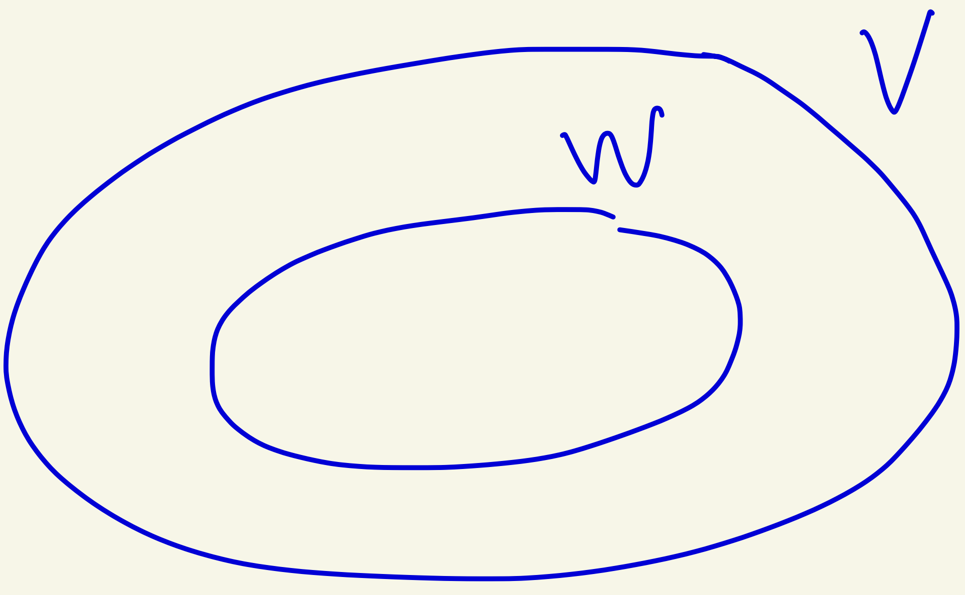
(b) If $m < n$, then $v_1, v_2, \ldots, v_m$ do not span V.

(c) If $m = n$ and $v_1, v_2, \ldots, v_m$ span V, then $v_1, v_2, \ldots, v_m$ are also linearly independent and hence form a basis for V.

(d) If $m = n$ and $v_1, v_2, \ldots, v_m$ are linearly independent, then $v_1, v_2, \ldots, v_m$ span V and hence form a basis for V.

(2) Let W be a subspace of V. Then W is finite-dimensional and $\dim(W) \leq \underbrace{n}_{\dim(V)}$

Moreover, $W = V$ if and only if $\dim(W) = \dim(V)$.



Proof: We have that $\dim(V) = n$.

① Let $v_1, v_2, \ldots, v_m \in V$.

(a) Suppose that $m > n$.

Since $\dim(V) = n$ we know that V has a basis with n vectors.

So, V is spanned by n vectors.

From a previous theorem, since $m > n$ we know that $v_1, v_2, \ldots, v_m$ are linearly dependent.

(b) Suppose $m < n$.

Let's show that $v_1, v_2, \ldots, v_m$ do not span V.

Suppose instead that $v_1, v_2, \ldots, v_m$ did span V.

Then from our previous results, since $m<n$, and $v_1, v_2, \ldots, v_m$ span V, we would have that any set of n vectors must be linearly dependent.

But since $\dim(V)=n$ there must be a basis for V of size n.

So, there is a set of n vectors in V that are linearly independent.

Contradiction.

So, $v_1, v_2, \ldots, v_m$ do not span V.

(c) Suppose $m=n$ and $v_1, v_2, \ldots, v_m$ span V

We want to show that $v_1, v_2, \ldots, v_m$ are linearly independent.

HW 2 - #7b

Suppose $V \neq \{\vec{0}\}$ is spanned by some finite set S of vectors. Prove that some subset of S is a basis for V

Let $S = \{v_1, v_2, \ldots, v_m\}$.

By this HW problem, there is a subset S' of S that is a basis for V.

Since $\dim(V) = n$, every basis for V has n vectors in it.

So, S' has $m=n$ vectors.

Thus, $S' = S$. Thus, $S = \{v_1, v_2, \ldots, v_m\}$ is a basis for V and is thus linearly independent.

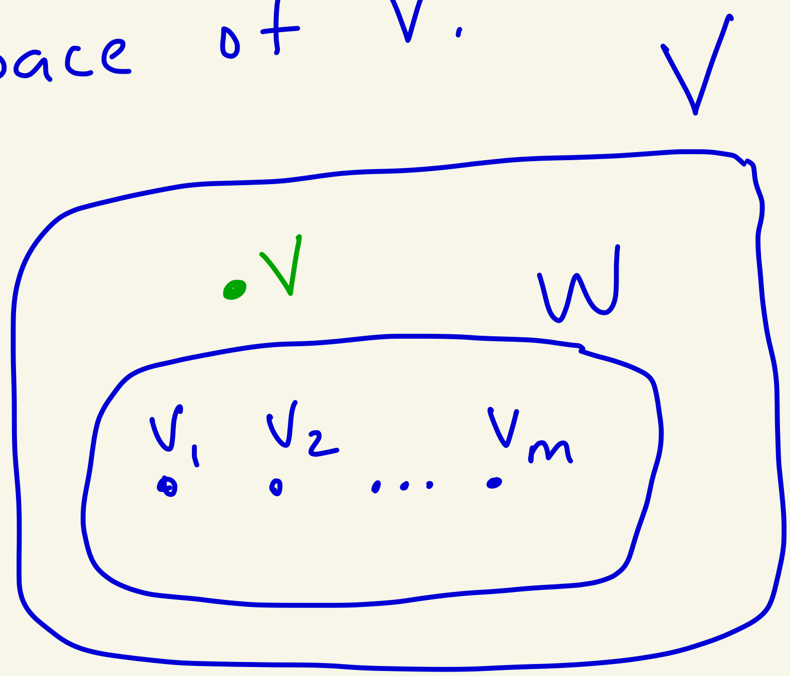
(d) Suppose $m = n = \dim(V)$ and $v_1, v_2, \ldots, v_m$ are linearly independent.

We want to show that $v_1, v_2, \ldots, v_m$ span V and hence are a basis for V.

Let $W = \text{span}(\{v_1, v_2, \ldots, v_m\})$.

So W is a subspace of V.

We will now show that $W = V$.



We know $W \subseteq V$.

We need to show that $V \subseteq W$.

Let $v \in V$.

Since $\dim(V) = n = m$ we know that the $n+1 = m+1$ vectors $v_1, v_2, \ldots, v_m, v$ are linearly dependent from part (a).

Thus, there exist
$c_1, c_2, \ldots, c_m, c_{m+1} \in F$,
not all equal to zero, where
$$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m + c_{m+1} v = \vec{0}$$

If $c_{m+1} = 0$, then
$$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m = \vec{0}$$
with not all $c_1, c_2, \ldots, c_m$ equalling zero.

But this would contradict the fact that $v_1, v_2, \ldots, v_m$ are linearly independent.

Thus, $c_{m+1} \neq 0$.

So, we can solve for v in
$$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m + c_{m+1} v = \vec{0}$$

and we get

$$v = \underbrace{c_{m+1}^{-1}}_{\substack{\text{exists} \\ \text{since } c_{m+1} \neq 0}} \left(-c_1 v_1 - c_2 v_2 - \dots - c_m v_m\right)$$

So,

$$v = \left(-c_{m+1}^{-1} c_1\right) v_1 + \left(-c_{m+1}^{-1} c_2\right) v_2 + \dots + \left(-c_{m+1}^{-1} c_m\right) v_m$$

Thus, $v \in \operatorname{span}(\{v_1, v_2, \dots, v_m\}) = W$.

So, $V = W$ and $v_1, v_2, \dots, v_m$ span V and are thus a basis for V. $\boxed{1}$

Now for part 2.

2

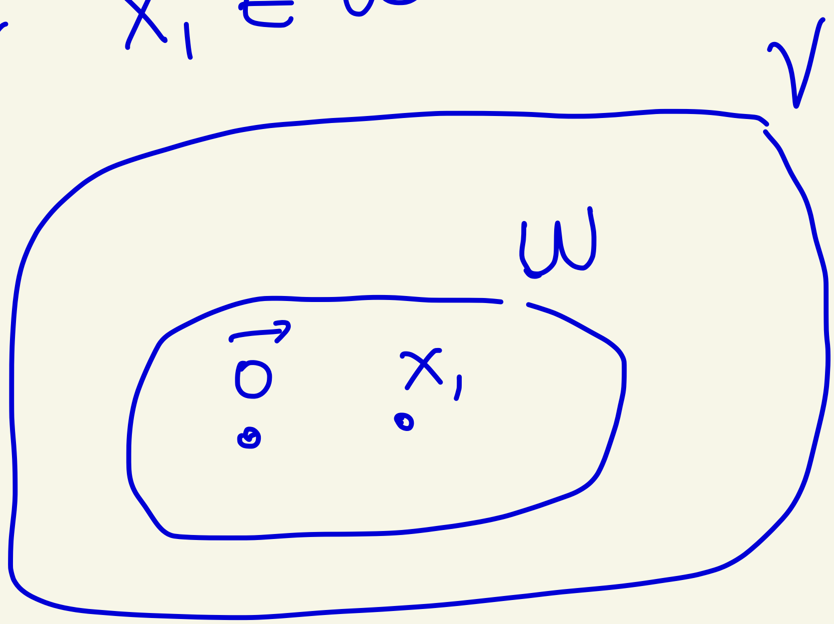
Let W be a subspace of V.

We first will show that W is finite-dimensional and $\dim(W) \leq n = \dim(V)$.

If $W = \{\vec{0}\}$, then W is finite-dimensional and $\dim(W) = 0 < n = \dim(V)$.

Now suppose $W \neq \{\vec{0}\}$.

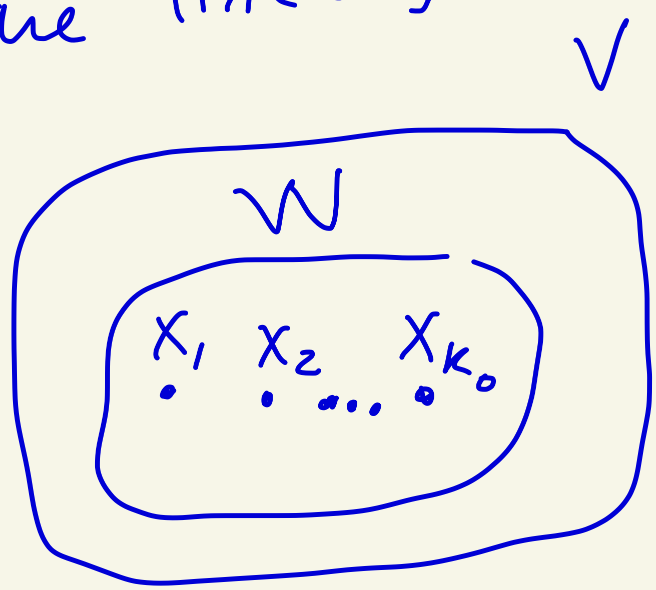
Then there exists $x_1 \in W$ with $x_1 \neq \vec{0}$.



Then, $\{x_1\}$ is a linearly independent set of vectors.

Because if $c_1 x_1 = \vec{0}$ then $c_1 = 0$ because $x_1 \neq \vec{0}$.

Continue to add vectors from W to this set such that at each stage k, the vectors $\{x_1, x_2, \ldots, x_k\}$ are linearly independent.



Since $W \subseteq V$ and $\dim(V) = n$, by part (a), there must reach a stage $k_0 \leq n$ where $S_0 = \{x_1, x_2, \ldots, x_{k_0}\}$ is linearly independent but adding any new vector from W to S_0 will yield a linearly dependent set.

HW 2 – 7(a)

Let S be a finite set of linearly independent vectors from V and let $x \in V$ with $x \notin S$. Then $S \cup \{x\}$ is linearly dependent iff $x \in \operatorname{span}(S)$

Let $x \in W$.

If $x \in S_0$, then $x \in \operatorname{span}(S_0)$.

If $x \notin S_0$, then by the construction of S_0 we have that $S_0 \cup \{x\}$ is linearly dependent. So by HW 2, 7(a), $x \in \operatorname{span}(S_0)$.

Thus, if $x \in W$, then $x \in \operatorname{span}(S_0)$.

So, $W = \operatorname{span}(S_0)$.

Since S_0 is a lin. ind. set, S_0 is a basis for W. Thus,

$\dim(W) = k_0 \leq n = \dim(V)$.

Now we show that $W = V$ iff $\dim(W) = \dim(V)$.

($\Rightarrow$) If $V = W$, then $\dim(V) = \dim(W)$.

($\Leftarrow$) Now suppose $\dim(W) = \dim(V)$.
Let's show that $W = V$.
Then W has a basis of $n = \dim(V)$ elements, call it $\beta = \{w_1, w_2, \ldots, w_n\}$

So, $W = \text{span}(\beta)$.

By part 1(d), since β is a set of n vectors that are linearly independent and $n = \dim(V)$, they must span V also!

So, β is a basis for V.

Thus, $W = \text{span}(\beta) = V$. $\boxed{}$