Ex: (Last time)

$V = F^n$, F is a field, F is the scalars

$$\boxed{\begin{array}{l} \text{standard basis} \\ \hline v_1 = (1, 0, \dots, 0) \\ v_2 = (0, 1, \dots, 0) \\ \quad \vdots \qquad\quad \vdots \\ v_n = (0, 0, \dots, 1) \end{array}}$$

v_i has a 1 in spot i and 0's everywhere else

So, $V = F^n$ has dimension n over F

Ex:

So, $\underbrace{\dim_{\mathbb{R}}(\mathbb{R}^3)}_{\text{means dimension of } V=\mathbb{R}^3 \text{ over } F=\mathbb{R}} = 3$

Standard basis

$v_1 = (1,0,0)$

$v_2 = (0,1,0)$

$v_3 = (0,0,1)$

__Ex:__ Let $F = \mathbb{R}$ or $F = \mathbb{C}$.

Let $V = P_n(F) = \{ a_0 + a_1 x + \cdots + a_n x^n \mid a_0, a_1, \ldots, a_n \in F \}$

$V = P_n(F)$ is a vector space over F.

One can show that

$$\left.\begin{array}{c} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{array}\right\} \begin{array}{l} n+1 \text{ vectors} \\ \text{called the standard basis} \end{array}$$

is a basis for $V = P_n(F)$ over F.

Thus,

$$\dim_F(P_n(F)) = \dim(P_n(F)) = n+1$$

this F just tells you what the scalars/field are/is

$\underline{\text{Ex:}}$ $V = P_2(\mathbb{R}) = \{ a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}$

$F = \mathbb{R}$

<u>standard basis</u>: $1, x, x^2$

So, $\underbrace{\dim(P_2(\mathbb{R}))}_{\substack{\text{or write} \\ \dim_{\mathbb{R}}(P_2(\mathbb{R})) \\ \text{to say what } F \text{ is}}} = 3$

Ex: $V = M_{m,n}(F) \leftarrow$ set of $m \times n$ matrices with entries from F

F is a field $\leftarrow$ field / scalars

One can show that $\dim(M_{m,n}(F)) = mn$

For example, $V = M_{3,2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \;\middle|\; a,b,c,d,e,f \in \mathbb{R} \right\}$

has standard basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$

So, $\dim(M_{3,2}(\mathbb{R})) = 6$

Ex: $V = M_{3,2}(\mathbb{R})$, $F = \mathbb{R}$

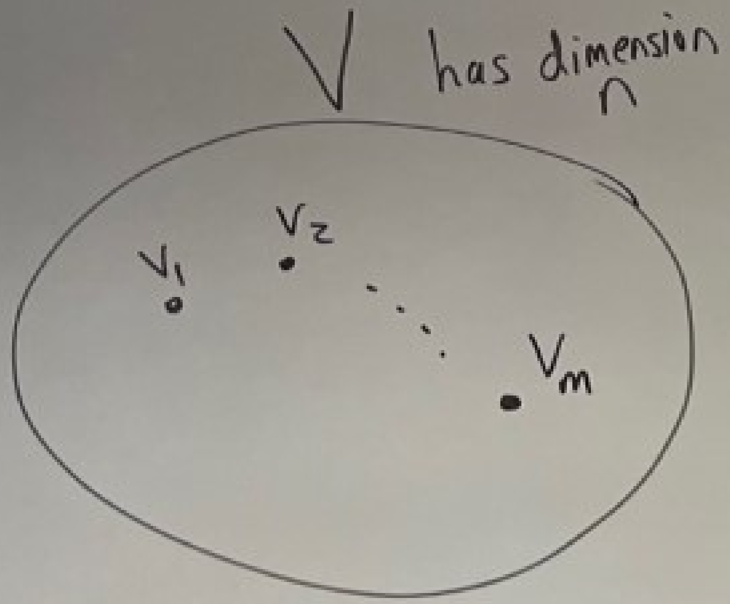
$$W = \operatorname{span}\left(\left\{\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}\right\}\right) = \left\{ a\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \,\middle|\, a \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} a & a \\ a & a \\ a & a \end{pmatrix} \,\middle|\, a \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -\pi & -\pi \\ -\pi & -\pi \\ -\pi & -\pi \end{pmatrix}, \cdots \right\}$$

W is a $\underbrace{\text{1-dimensional}}$ subspace of $V = M_{3,2}(\mathbb{R})$

basis: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$



Theorem: Let V be a vector space over a field F. Suppose V is finite dimensional with $\dim(V) = n > 0$.

Then:

1. Let $v_1, v_2, \ldots, v_m \in V$.

(a) If $m > n$, then $v_1, v_2, \ldots, v_m$ are linearly dependent.

(b) If $m < n$, then $v_1, v_2, \ldots, v_m$ do not span V.

(c) If $m = n$ and $v_1, v_2, \ldots, v_m$ are linearly independent, then $v_1, v_2, \ldots, v_m$ must span V (and hence are a basis for V).

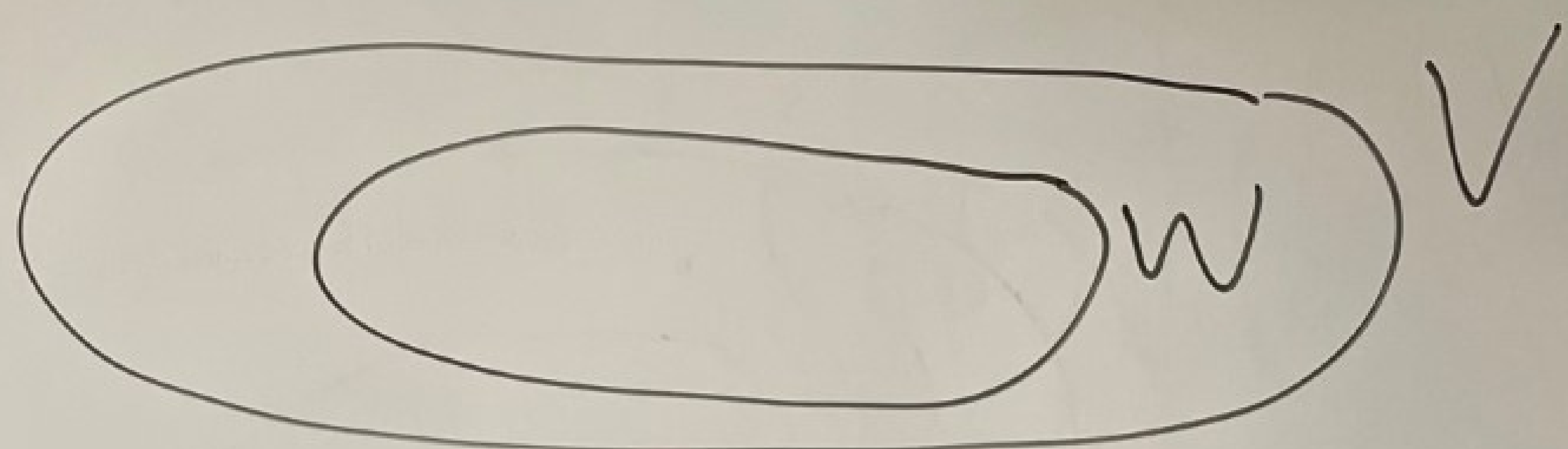
Will use definitely

(d) If $m = n$ and $v_1, v_2, \ldots, v_m$ span V, then $v_1, v_2, \ldots, v_m$ are linearly independent (and hence are a basis for V)

(2) Let W be a subspace of V.

Then, W is also finite dimensional and $\dim(W) \leq \underbrace{n}_{\dim(V)}$

More over, $W = V$ iff $\dim(W) = \dim(V)$.



Proof: I'll post it on the website under the notes for today.

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

What can you say about these vectors?

$v_1 = (1,0)$, $v_2 = (1,1)$, $v_3 = (0,2)$

$\dim(\mathbb{R}^2) = 2$

and we have 3 vectors.

Since $3 > 2$,

v_1, v_2, v_3 are lin. dep.

Ex: $V = P_2(\mathbb{R})$, $F = \mathbb{R}$

What can we say about these vectors?

$$\left.\begin{aligned} v_1 &= 1 - x + x^2 \\ v_2 &= 1 + x \end{aligned}\right\}$$

We know $\dim(P_2(\mathbb{R})) = 2+1 = 3$

And we have 2 vectors.

Since $2 < 3$, v_1, v_2 do not span $P_2(\mathbb{R})$.

Math 2550
HW 7
Part 1
#8(a)

Ex: $V = P_2(\mathbb{R})$, $F = \mathbb{R}$

Is $v_1 = 1$, $v_2 = 1+x$, $v_3 = 1+x+x^2$ a basis for $V = P_2(\mathbb{R})$ over $F = \mathbb{R}$?

We know $\dim(P_2(\mathbb{R})) = 2+1 = 3$.

We have 3 vectors.

So if we can show they are linearly independent, then by the thm ①(c) the vectors will also span and be a basis.

Let's find the solutions to

$$\underbrace{c_1(1)}_{v_1} + \underbrace{c_2(1+x)}_{v_2} + \underbrace{c_3(1+x+x^2)}_{v_3} = \underbrace{\vec{0}}_{0+0x+0x^2}$$

This becomes

$$c_1 + c_2 + c_2x + c_3 + c_3x + c_3x^2 = 0 + 0x + 0x^2$$

This becomes

$$(c_1+c_2+c_3) + (c_2+c_3)x + c_3x^3 = 0 + 0x + 0x^2$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}}$$

this system is already row-reduced

$$\boxed{c_1} + c_2 + c_3 = 0$$
$$\boxed{c_2} + c_3 = 0$$
$$\boxed{c_3} = 0$$

leading variables: c_1, c_2, c_3

free variables: none

Solve for leading variables:

$$\begin{aligned} c_1 &= -c_2 - c_3 \quad (1) \\ c_2 &= -c_3 \quad (2) \\ c_3 &= 0 \quad (3) \end{aligned}$$

Back substitute:

(3) $c_3 = 0$

(2) $c_2 = -c_3 = -0 = 0$

(1) $c_1 = -c_2 - c_3 = -0 - 0 = 0$

So the only solution is $c_1 = c_2 = c_3 = 0$.

Thus, $v_1 = 1, v_2 = 1+x, v_3 = 1+x+x^2$ are lin. ind.

So, by thm (1)(c) they form a basis for $V = P_2(\mathbb{R})$.

2550 - HW 7 - Part 2

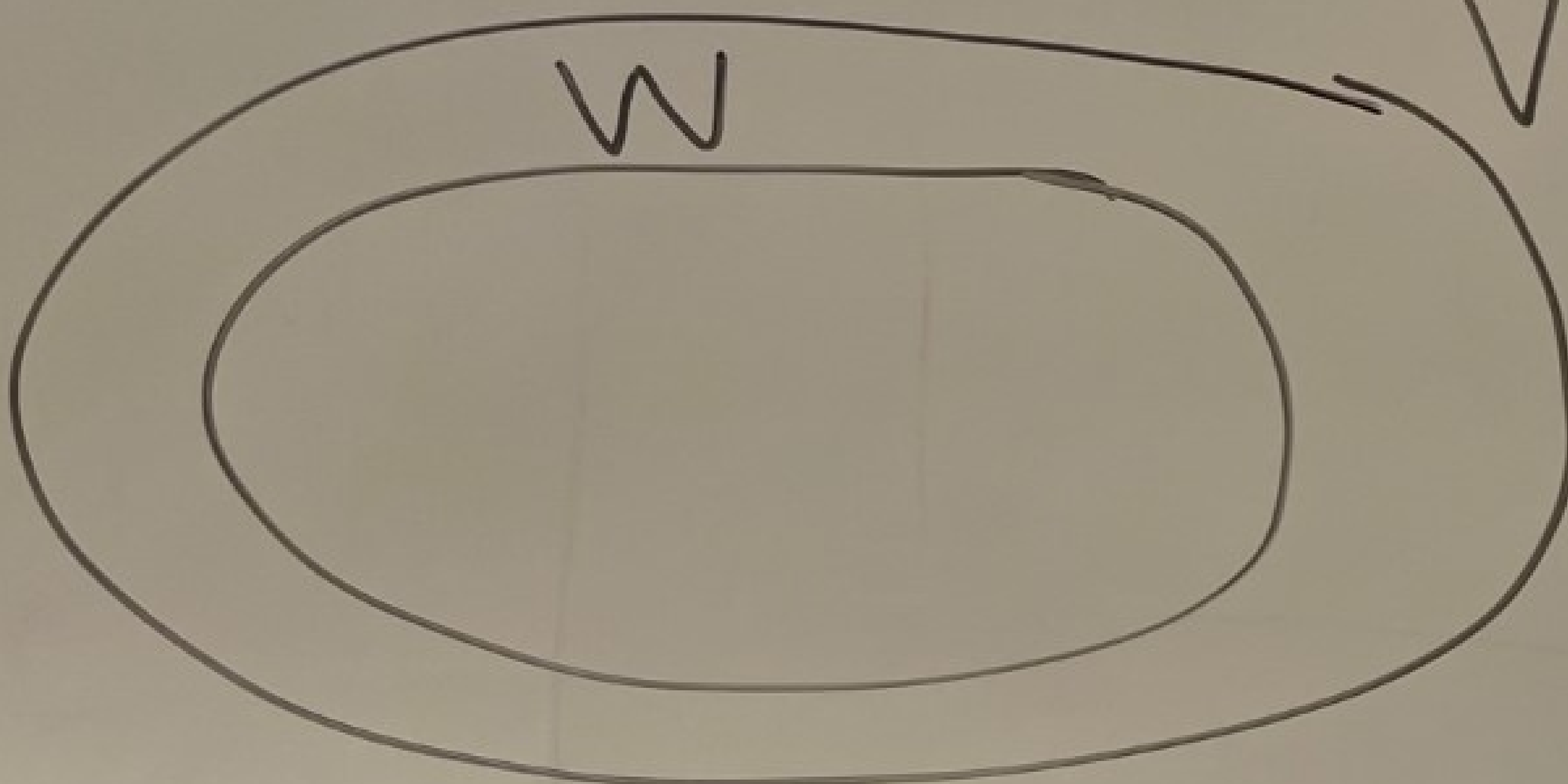
(2) Let $V = M_{2,2}(\mathbb{R})$, $F = \mathbb{R}$.

Let

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{matrix} a+b+c=0 \\ a,b,c,d \in \mathbb{R} \end{matrix} \right\} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}, \dots \right\}$$

You can prove W is a subspace of $V = M_{2,2}(\mathbb{R})$. [Try it]

What is $\dim(W)$?



$V = M_{2,2}(\mathbb{R})$

$\uparrow$

$\dim(V) = 4$

So by theorem 2,

$\dim(W) \leq 4$.

First let's get a spanning set for W

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$.

Then, $a + b + c = 0$.

W

Think of $\boxed{a}+\boxed{b}+\boxed{c}=0$ as a system.

leading variable

free variables (and d also)

So, $a=-b-c$ and b,c can be anything. And d can be anything also.

So,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -b-c & b \\ c & d \end{pmatrix} = \begin{pmatrix} -b & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -c & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$

$$= b\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So,

$$W = \operatorname{span}\left(\left\{\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}\right)$$

Let's continue this next time...