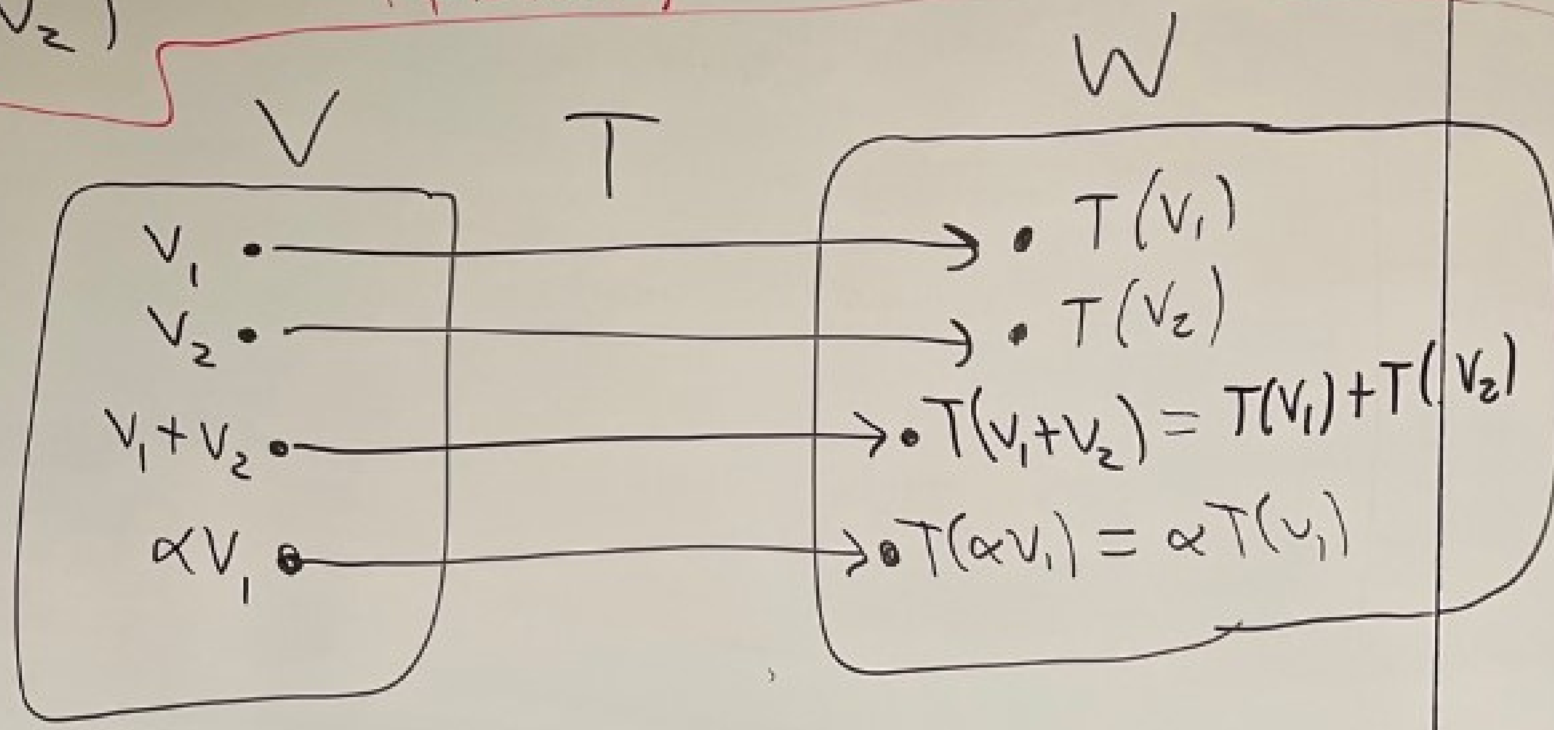
HW 3 – LINEAR TRANSFORMATIONS

Definition: Let V and W be vector spaces over a field F. Let $T: V \to W$. We say that T is a <u>linear transformation</u> if for every $v_1, v_2 \in V$ and $\alpha \in F$ we have

① $T(v_1 + v_2) = T(v_1) + T(v_2)$

and ② $T(\alpha v_1) = \alpha T(v_1)$

picture



You can say:

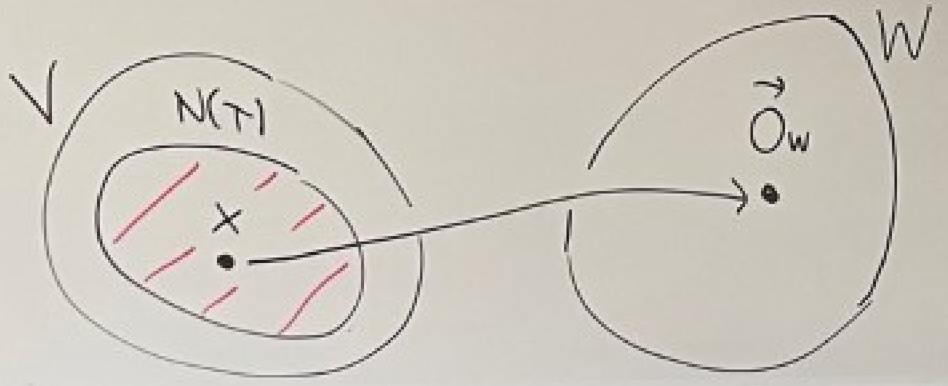
T "preserves" vector addition and scalar mult.

You can combine ① and ② and replace it with:

$$\boxed{T(\alpha v_1+\beta v_2)=\alpha T(v_1)+\beta T(v_2) \text{ for all } v_1, v_2 \in V \text{ and } \alpha,\beta \in F}$$

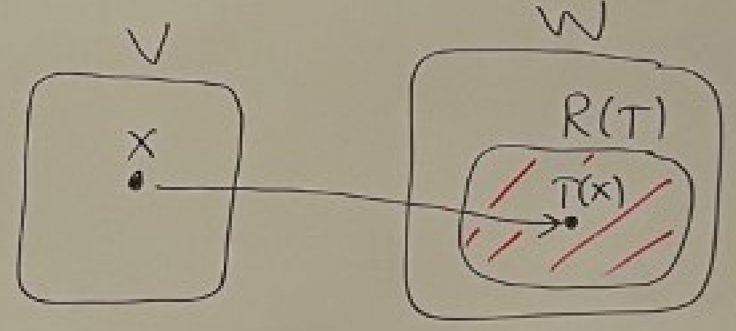
Let $\vec{0}_w$ be the zero vector of W.
We define the <u>nullspace</u> (or the <u>kernel</u>) of T to be

$$N(T)=\{x\in V \mid T(x)=\vec{0}_w\}$$



The <u>range</u> (or <u>image</u>) of T is

$$R(T)=\{T(x) \mid x\in V\}$$



<u>Note:</u> We will show later that
$N(T)$ is a subspace of V
and
$R(T)$ is a subspace of W.

If $N(T)$ is finite dimensional then we call the dimension of $N(T)$ the <u>nullity</u> of T and write

$$\text{nullity}(T)=\dim(N(T)).$$

If $R(T)$ is finite dimensional then we call the dimension of $R(T)$ the <u>rank</u> of T and write

$$\text{rank}(T)=\dim(R(T)).$$

Ex: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x,y,z) = (x,y)$.

Here $V = \mathbb{R}^3$, $W = \mathbb{R}^2$, $F = \mathbb{R}$.

Some example computations: $T(1,0,\pi) = (1,0)$

$$T\left(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}\right) = \left(\tfrac{1}{2}, \tfrac{1}{3}\right)$$

Note: T is like a matrix

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{T} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

T is a linear transformation!

proof: Let's do the 1-part version.

Let $\alpha, \beta \in \mathbb{R}$ and $v_1, v_2 \in \mathbb{R}^3$.

So, $v_1 = (a,b,c)$, $v_2 = (d,e,f)$ where $a,b,c,d,e,f \in \mathbb{R}$.

Thus,

$$\begin{aligned}
T(\alpha v_1 + \beta v_2) &= T\big(\alpha(a,b,c) + \beta(d,e,f)\big) \\
&= T\big((\alpha a, \alpha b, \alpha c) + (\beta d, \beta e, \beta f)\big) \\
&= T(\alpha a + \beta d, \alpha b + \beta e, \alpha c + \beta f) \\
&\overset{*}{=} (\alpha a + \beta d, \alpha b + \beta e) \\
&= (\alpha a, \alpha b) + (\beta d, \beta e) \\
&= \alpha(a,b) + \beta(d,e) \overset{*}{=} \alpha T(a,b,c) + \beta T(d,e,f) = \alpha T(v_1) + \beta T(v_2)
\end{aligned}$$

$T(x,y,z) = (x,y)$

$\blacksquare$

atrix

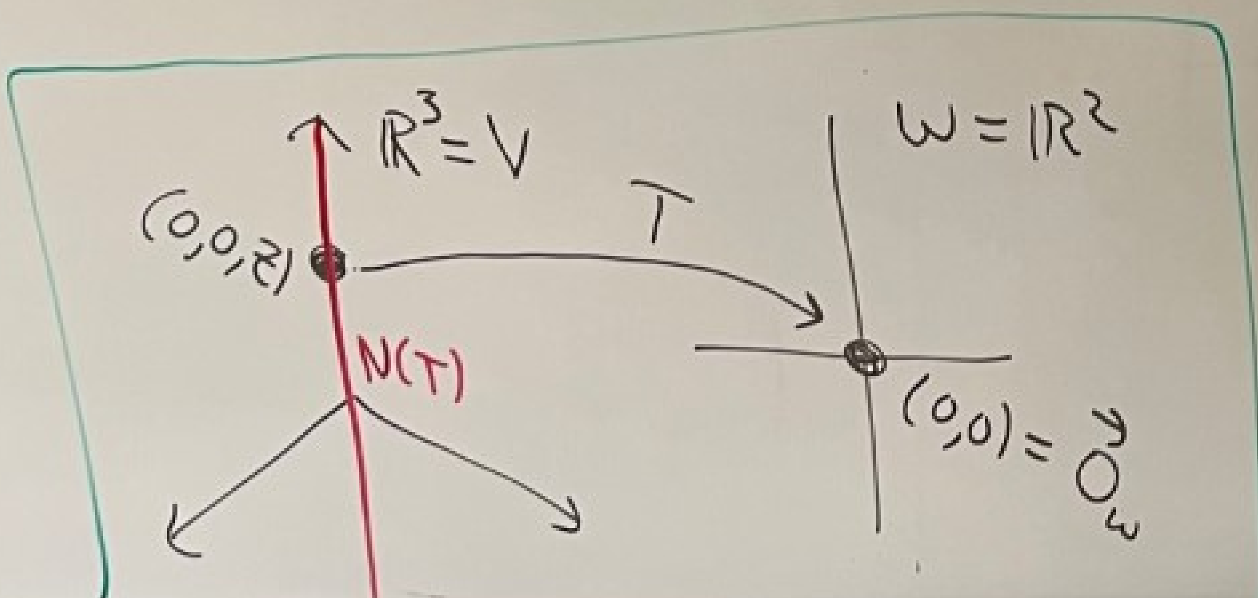
Nullspace of T

$$N(T)=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \,\middle|\, T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\right\}$$

$$=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \,\middle|\, \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\right\}$$

$$=\left\{\begin{pmatrix}0\\0\\z\end{pmatrix} \,\middle|\, z\in\mathbb{R}\right\}$$

$$=\left\{z\begin{pmatrix}0\\0\\1\end{pmatrix} \,\middle|\, z\in\mathbb{R}\right\}=\operatorname{span}\left(\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\right)$$



HW: A single non-zero vector is linearly independent

So a basis for $N(T)$ is $\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

Thus,

$$\text{nullity}(T)=\dim(N(T))=1$$

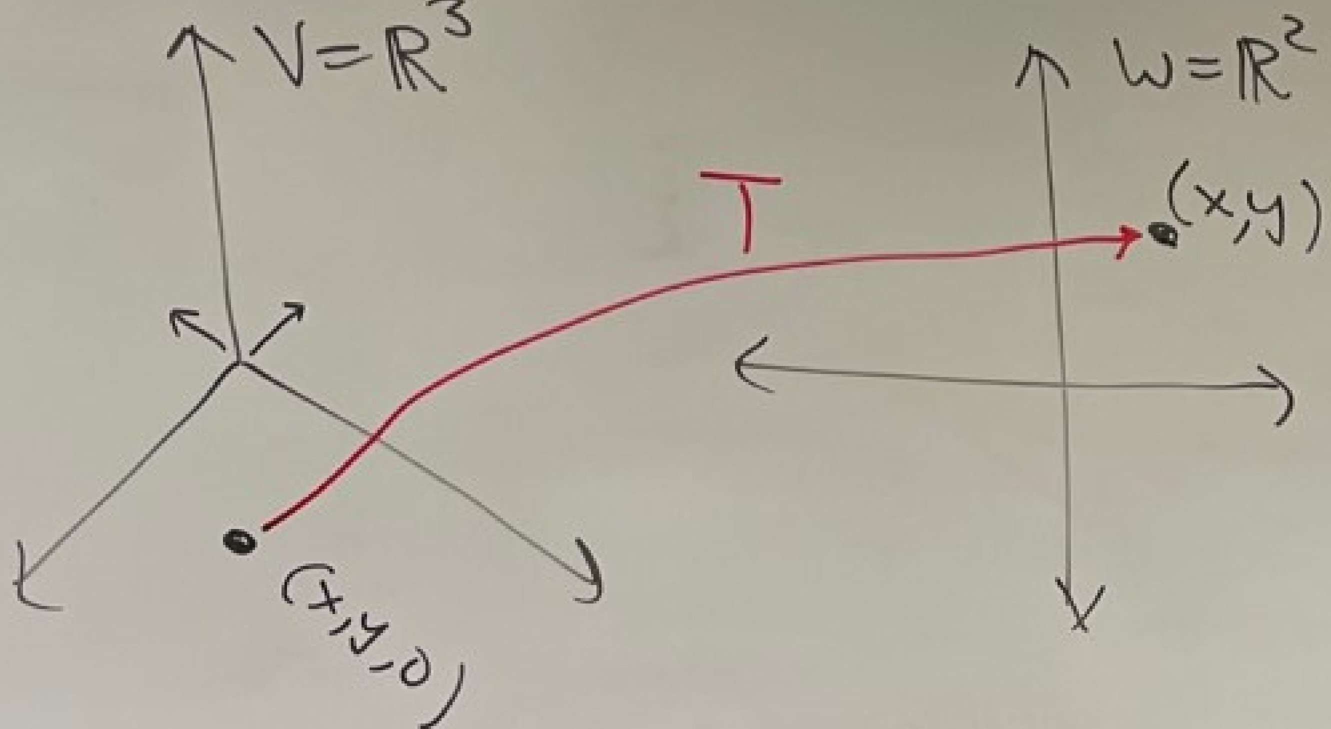
Range of T

T is onto $\mathbb{R}^2$.

Why?

Given $(x,y) \in \mathbb{R}^2$,

then $T(x,y,0) = (x,y)$



So,

$$\underbrace{R(T)}_{\text{range of } T} = \mathbb{R}^2.$$

So then

$\text{rank}(T) = \dim(R(T)) = 2.$

Note:

$$3 = 1 + 2$$

$\dim(V) = \text{nullity}(T) + \text{rank}(T)$

$V = \mathbb{R}^3$

Ex: Let $n \geq 1$ be an integer.

Define $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$

by $T(f) = \underbrace{f'}_{\text{derivative of } f}$

T is linear:

Let $f_1, f_2 \in P_n(\mathbb{R})$ and $\alpha \in \mathbb{R}$.

① $T(f_1+f_2) = (f_1+f_2)' \overset{\text{Calculus}}{=} f_1' + f_2' = T(f_1) + T(f_2)$

and ② $T(\alpha f_1) = (\alpha f_1)' \overset{\text{Calculus}}{=} \alpha(f_1') = \alpha T(f_1)$ $\boxtimes$

Nullspace of T

$$N(T) = \{a_0 + a_1 x + \cdots + a_n x^n \mid T(a_0 + a_1 x + \cdots + a_n x^n) = \vec{0}\}$$

$$= \{a_0 + a_1 x + \cdots + a_n x^n \mid a_1 + 2a_2 x + \cdots + n a_n x^{n-1} = \vec{0}\}$$

$$= \{a_0 + a_1 x + \cdots + a_n x^n \mid a_1 = a_2 = \cdots = a_n = 0\}$$

$$= \{a_0 \mid a_0 \in \mathbb{R}\} = \{a_0 \cdot 1 \mid a_0 \in \mathbb{R}\}$$

$$= \mathrm{span}(\{1\})$$

A basis for $N(T)$ is $\{1\}$.

Thus, $\mathrm{nullity}(T) = \dim(N(T)) = 1$.

Range of T

next time...