HW 1 Solutions

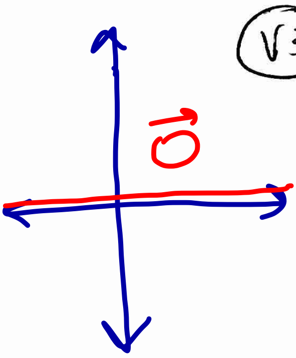
① (a) $V = C(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous on all of } \mathbb{R} \}$ is a vector space.

pf:

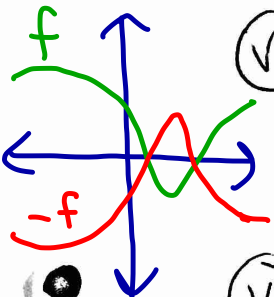
First note that if $f, g \in C(\mathbb{R})$ and $\alpha \in \mathbb{R}$ then $f+g$ and αf are in $C(\mathbb{R})$ since adding continuous functions and multiplying a continuous function by a real number produces continuous functions.

(V1) If $f, g \in C(\mathbb{R})$ then $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$. So $f+g = g+f$.

(V2) If $f, g, h \in C(\mathbb{R})$, then $[f + (g+h)](x) = f(x) + (g+h)(x) = f(x) + g(x) + h(x) = (f+g)(x) + h(x) = [(f+g)+h](x)$.
So, $f + (g+h) = (f+g) + h$.



(V3) Define $\vec{0}$ as $\vec{0}(x) = 0$ for all $x \in \mathbb{R}$. Then if $f \in C(\mathbb{R})$ we have that $(f + \vec{0})(x) = f(x) + \vec{0}(x) = f(x)$ and $(\vec{0} + f)(x) = \vec{0}(x) + f(x) = f(x)$. So, $\vec{0}$ is a zero vector of $C(\mathbb{R})$.



(V4) Given $f \in C(\mathbb{R})$, define $(-f)(x) = -f(x)$.
Then $(f + (-f))(x) = f(x) - f(x) = 0$ and $((-f) + f)(x) = -f(x) + f(x) = 0$. So, $f + (-f) = \vec{0}$ and $(-f) + f = \vec{0}$.

(V5) If $f \in C(\mathbb{R})$, then $1 \cdot f(x) = f(x)$.
So, $1 \cdot f = f$.

(V6) If $f \in C(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then $[(\alpha\beta) f](x) = (\alpha\beta) f(x) = \alpha(\beta f(x)) = [\alpha(\beta f)](x)$.
So, $(\alpha\beta) \cdot f = \alpha \cdot (\beta \cdot f)$

(V7) Let $f, g \in C(\mathbb{R})$ and $\alpha \in \mathbb{R}$, then $[\alpha \cdot (f+g)](x) =$
$\alpha[(f+g)(x)] = \alpha[f(x)+g(x)] = \alpha f(x) + \alpha g(x)$. So, $\alpha \cdot (f+g) = \alpha \cdot f + \alpha \cdot g$

(V8) Let $f \in C(\mathbb{R})$, and $\alpha, \beta \in \mathbb{R}$. Then $(\alpha+\beta) f(x) = \alpha f(x) + \beta f(x)$
$= (\alpha \cdot f)(x) + (\beta \cdot f)(x)$
So, $(\alpha+\beta) \cdot f = \alpha \cdot f + \beta \cdot f$

[redacted]

1 (b) $V = \mathbb{R}^2$ is not a vector space over $F = \mathbb{R}$ using the operations

$$(x,y) + (a,b) = (x+a, y+b)$$

$$\alpha \odot (x,y) = (2\alpha x, 2\alpha y)$$

For example VS fails.

The 1 element of $F = \mathbb{R}$ is 1.

And,

$$1 \odot (5,3) = (2 \cdot 1 \cdot 5, 2 \cdot 1 \cdot 3)$$
$$= (10, 6)$$

So,

$$1 \odot (5,3) \neq (5,3)$$

So, VS $1 \odot w = w \quad \forall w \in \mathbb{R}^2$ fails to be true.

① (c) $V = \{x \in \mathbb{R} \mid x > 0\}$ is a vector space over $F = \mathbb{R}$

using

$$x \oplus y = xy$$

and

$$\alpha \odot x = x^{\alpha}$$

proof:

Note that if $x, y \in V$ and $\alpha \in \mathbb{R}$, then $x, y \in \mathbb{R}$ with $x > 0, y > 0$.

Thus, $x \oplus y = xy \in \mathbb{R}$ with $x \oplus y = xy > 0$.

So, $x \oplus y \in V$.

Also, $\alpha \odot x = x^{\alpha} \in \mathbb{R}$ and $x^{\alpha} > 0$.

So, $\alpha \odot x \in V$.

(V1) Let $x, y \in V$. Then, $x \oplus y = xy =$

$= yx = y \oplus x$

(V2) Let $x, y, z \in V$. Then,

$$x \oplus (y \oplus z) = x \oplus yz = x(yz) = (xy)z$$
$$= (xy) \oplus z = (x \oplus y) \oplus z.$$

(V3) Let $\vec{0} = 1$. Note that $1 > 0$ and so $1 \in V$. Also, if $x \in V$, then

$$x \oplus 1 = x \cdot 1 = x = 1 \cdot x = 1 \oplus x$$

Thus, 1 is the zero vector $\vec{0}$ in V.

(V4) Let $w \in V$. So, $w \in \mathbb{R}$ with $w > 0$. Thus, $z = \frac{1}{w} \in \mathbb{R}$ with $z = \frac{1}{w} > 0$. So, $z = \frac{1}{w} \in V$.

And,

$$w \oplus z = w \cdot \frac{1}{w} = 1 = \frac{1}{w} \cdot w = z \oplus w$$

the "zero" vector $\vec{0}$ in V

(V5) Let $w \in V$. Then $w \in \mathbb{R}$ and $w > 0$.

The 1 element in $F = \mathbb{R}$ is 1.

So,

$$1 \odot w = w^1 = w.$$

(V6) Let $w \in V$ and $a, b \in F = \mathbb{R}$.

Then
$$\begin{aligned}(ab) \odot w &= w^{ab} \\ &= w^{ba} \\ &= (w^b)^a \\ &= a \odot w^b \\ &= a \odot (b \odot w)\end{aligned}$$

(V7) Let $v_1, v_2 \in V$ and $a \in \mathbb{R}$.

Then
$$\begin{aligned}a \odot (v_1 \oplus v_2) &= a \odot (v_1 v_2) = (v_1 v_2)^a \\ &= v_1^a v_2^a = v_1^a \oplus v_2^a \\ &= (a \odot v_1) \oplus (a \odot v_2)\end{aligned}$$

V8 Let $a, b \in \mathbb{R}$ and $w \in V$.

Then,

$$(a+b) \odot w = w^{a+b} = w^a w^b$$

$$= w^a \oplus w^b$$

$$= (a \odot w) \oplus (b \odot w)$$

the + in $F = \mathbb{R}$ is the normal +

[redacted]

(2) (a) $W = \{(a,b,c) \mid a = 3b \text{ and } c = -b\}$ is a subspace of $\mathbb{R}^3$.

proof:

(1) $(a,b,c) = (0,0,0)$ satisfies $a = 3b$ and $c = -b$.

(2) Let $(a_1,b_1,c_1), (a_2,b_2,c_2) \in W$. Then $a_1 = 3b_1$ and $c_1 = -b_1$ and $a_2 = 3b_2$ and $c_2 = -b_2$. And so, $a_1 + a_2 = 3(b_1 + b_2)$ and $c_1 + c_2 = -(b_1 + b_2)$. Thus, $(a_1,b_1,c_1) + (a_2,b_2,c_2) = (a_1+a_2, b_1+b_2, c_1+c_2) \in W$.

(3) Let $(a,b,c) \in W$ and $\alpha \in \mathbb{R}$. Then $a = 3b$ and $c = -b$. So, $\alpha a = 3\alpha b$ and $\alpha c = -(\alpha b)$. Thus, $\alpha(a,b,c) = (\alpha a, \alpha b, \alpha c) \in W$.

(b) $V = \mathbb{C}^2$, $F = \mathbb{C}$

$$W = \{(a,b,c) \mid a = c+2\}$$

W is not a subspace of V.
For example $(a,b,c) = (0,0,0)$ does not satisfy $a = c+2$ since $0 \neq 0+2$.

(c) $V = M_{2,2}(\mathbb{R})$, $F = \mathbb{R}$

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a+b+c+d = 0 \right\}.$$

W is a subspace of V.

- $\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$ since it satisfies $a+b+c+d=0$
- Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in W$. Then $a+b+c+d=0$ and $e+f+g+h=0$. So, $(a+e)+(b+f)+(c+g)+(d+h)=0$ and so $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$ is in W.
- Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$ and $\alpha \in \mathbb{R}$. Then $a+b+c+d=0$. Thus $(\alpha a)+(\alpha b)+(\alpha c)+(\alpha d)=0$. So, $\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$ is in W.

(d) $V = M_{2,2}(\mathbb{R})$, $F = \mathbb{R}$, $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad-bc=0 \right\}$.

W is not a subspace of V.
It is true that $\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$ and if $\alpha \in \mathbb{R}$ and $M \in W$, then $\alpha M \in W$. However W is not closed under addition. For example, $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix}$ are both in W but $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \notin W$ since $1 \cdot 4 - 2 \cdot 3 = -2 \neq 0$.

(e) $V = P_2(\mathbb{R})$, $F = \mathbb{R}$, $W = \{a+bx+cx^2+dx^3 \mid a=0\}$.

W is a subspace of V.

- $\vec{0} = 0+0x+0x^2+0x^3 \in W$ since $a=0$ for this vector.

- Let $f_1 = 0+bx+cx^2+dx^3$, $f_2 = 0+ex+fx^2+gx^3$ be in W. Then $f_1+f_2 = 0+(b+e)x+(c+f)x^2+(d+g)x^3$ is in W since its constant term is 0.
- Let $\alpha \in \mathbb{R}$ and $f = 0+bx+cx^2+dx^3 \in W$. Then $\alpha f = 0+(\alpha b)x+(\alpha c)x^2+(\alpha d)x^3 \in W$.

(f) $V = P_3(\mathbb{C})$, $F = \mathbb{C}$,
$W = \{a+bx \mid a+b = i\}$

W is not a subspace.
For example, $\vec{0} = 0+0x+0x^2+0x^3$ does not satisfy $a+b=i$.
W is not closed under addition also. For example $(-1)+(1+i)x$ and $0+ix$ are both in W but their sum is $(-1)+(1+2i)x$ isn't in W since $(-1)+(1+2i) = 2i \neq i$.

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(a) Suppose that $0_1, 0_2 \in F$ with $0_1 + a = a + 0_1 = a$ and $0_2 + a = a + 0_2 = a$ for all $a \in F$. Then

$$0_1 = 0_1 + 0_2 = 0_2 .$$

since $a = a + 0_2$ $\forall a \in F$

since $0_1 + a = a$ $\forall a \in F$

(b) Suppose that $1_1, 1_2 \in F$ with $1_1 \cdot a = a \cdot 1_1 = a$ and $1_2 \cdot a = a \cdot 1_2 = a$ for all $a \in F$. Then

$$1_1 = 1_1 \cdot 1_2 = 1_2 .$$

since $a = a \cdot 1_2$ $\forall a \in F$

since $1_1 \cdot a = a$ $\forall a \in F$

(c) Let $a \in F$ and $d_1, d_2 \in F$ with $a + d_1 = d_1 + a = 0$ and $a + d_2 = d_2 + a = 0$ for all $a \in F$.

Thus, $a + d_1 = 0 = a + d_2$.

Now add d_1 to both sides to get $d_1 + (a + d_1) = d_1 + (a + d_2)$

So, $(d_1 + a) + d_1 = (d_1 + a) + d_2$. Thus, $0 + d_1 = 0 + d_2$.

Hence $d_1 = d_2$.

(d) Let $b \in F$ with $b \neq 0$ and $bf_1 = f_1 b = 1$ and $bf_2 = f_2 b = 1$ for some $f_1, f_2 \in F$.

Then $bf_1 = bf_2$. So, $f_1(bf_1) = f_1(bf_2)$.

Thus, $(f_1 b) f_1 = (f_1 b) f_2$. So, $1f_1 = 1f_2$.

Hence $f_1 = f_2$.

④

(a) Note that $0x = (0+0)x = 0x + 0x$. Now add $-0x$ to both sides to get $-0x + 0x = -0x + 0x + 0x$.

So, $\vec{0} = 0x$.

(b) Note that

$$(-a)x + ax = (-a+a)x = 0x \overset{\boxed{\text{part (a)}}}{=} \vec{0} \quad \text{and}$$

$$ax + (-a)x = (a-a)x = 0x = \vec{0}.$$

So, $(-a)x = -(ax)$.

Also,

$$a(-x) + ax = a(x-x) = a\vec{0} \overset{\boxed{\text{part(c)}}}{=} \vec{0}$$

and

$$ax + a(-x) = a(x-x) = a\vec{0} = \vec{0}$$

So, $a(-x) = -(ax)$.

Hence $(-a)x = -(ax) = a(-x)$.

(c) We have that

$$a\vec{0} = a(\vec{0} + \vec{0}) = a\vec{0} + a\vec{0}.$$

Add $-(a\vec{0})$ to both sides to get

$$-(a\vec{0}) + a\vec{0} = -(a\vec{0}) + a\vec{0} + a\vec{0}.$$

So, $\vec{0} = a\vec{0}$.

(5)

($\Rightarrow$) Suppose W is a subspace of V. Then there exists $\vec{0}_W \in W$ where $\vec{0}_W + x = x + \vec{0}_W = x$ for all $x \in W$ since W is a vector space. Since $\vec{0}_V$ also satisfies the same equation, $\vec{0}_V$ is also a zero vector for W. Thus $\vec{0}_W = \vec{0}_V \in W$ since the zero vector of a vector space is unique.

Since W is a vector space, by definition, $x+y \in W$ and $cx \in W$ for all $x, y \in W$ and $c \in F$.

($\Leftarrow$) Now suppose W is a subset of V such that (a) $\vec{0} \in W$, (b) $x+y \in W$ for all $x,y \in W$, and (c) $cx \in W$ for all $c \in F$, $x \in W$.

(V1), (V2), (V5), (V6), (V7), (V8) are all true for W since any equation that is true for all the elements of V must also be true for all the elements of W since $W \subseteq V$. (V3) is true for W since $\vec{0} \in W$. Now suppose $x \in W$. Since $x \in W$, by assumption $(-1)x \in W$. So, by 4(b), $-x = -(1x) = (-1)x \in W$. So, (V4) is true for W. Thus, W is a vector space and hence a subspace of V.

⑥ We use problem 5.

(a) Since W_1 and and W_2 are subspaces of V, we have that $\vec{0} \in W_1$ and $\vec{0} \in W_2$. So, $\vec{0} \in W_1 \cap W_2$.

(b) Let $x, y \in W_1 \cap W_2$

So, $x, y \in W_1$ and $x, y \in W_2$.

Since W_1 is a subspace, $x+y \in W_1$.

Since W_2 is a subspace, $x+y \in W_2$.

So, $x+y \in W_1 \cap W_2$.

(c) Let $c \in F$ and $x \in W_1 \cap W_2$.

So $x \in W_1$ and $x \in W_2$.

Since $c \in F$ and $x \in W_1$, we have $cx \in W_1$
since W_1 is a subspace.

Since $c \in F$ and $x \in W_2$, we have $cx \in W_2$
since W_2 is a subspace.

So, $cx \in W_1 \cap W_2$.

By (a), (b), (c), $W_1 \cap W_2$ is a subspace.

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(a) Let $x \in W_1$. Since $\vec{0} \in W_2$ we have that $x = x + \vec{0} \in W_1 + W_2$. Hence $W_1 \subseteq W_1 + W_2$.

Let $y \in W_2$. Since $\vec{0} \in W_1$ we have that $y = \vec{0} + y \in W_1 + W_2$. Hence $W_2 \subseteq W_1 + W_2$.

(b)
- Since W_1 and W_2 are subspaces, $\vec{0} \in W_1$ and $\vec{0} \in W_2$. So, $\vec{0} = \vec{0} + \vec{0} \in W_1 + W_2$.
- Let $a, b \in W_1 + W_2$. Then $a = x_a + y_a$ and $b = x_b + y_b$ where $x_a, x_b \in W_1$ and $y_a, y_b \in W_2$. Since W_1 is a subspace, $x_a + x_b \in W_1$. Since W_2 is a subspace, $y_a + y_b \in W_2$.

So,
$$a + b = (x_a + y_a) + (x_b + y_b)$$
$$= (x_a + x_b) + (y_a + y_b) \in W_1 + W_2.$$

- Let $e \in W_1 + W_2$ and $\alpha \in F$. Then $e = m + n$ where $m \in W_1$ and $n \in W_2$. Since W_1 is a subspace $\alpha m \in W_1$. Since W_2 is a subspace $\alpha n \in W_2$. So,

$$\alpha e = \alpha m + \alpha n \in W_1 + W_2.$$

(c) Suppose W is a subspace of V and $W_1 \subseteq W$, $W_2 \subseteq W$. Let $a \in W_1 + W_2$. Then $a = x + y$ for some $x \in W_1$ and $y \in W_2$. Since $x, y \in W$, $x + y \in W$. So, $a \in W$.

Thus, $W_1 + W_2 \subseteq W$.