Homework #2

①

(a) W_1 is a subspace of $M_{2,2}(\mathbb{R})$

- $\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$ for $a=b=c=0$. So, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W_1$.
- Let $x, y \in W_1$. Then $x = \begin{pmatrix} a_1 & b_1 \\ c_1 & a_1 \end{pmatrix}$ and $y = \begin{pmatrix} a_2 & b_2 \\ c_2 & a_2 \end{pmatrix}$ where $a_1, b_1, c_1 \in \mathbb{R}$. So, $x+y = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & a_1+a_2 \end{pmatrix}$, which is in W_1 as it is of the form $\begin{pmatrix} a & b \\ c & a \end{pmatrix}$.
- Let $x = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in W_1$ and $\alpha \in \mathbb{R}$. Then $\alpha x = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha a \end{pmatrix}$ is in W_1 since it is of the required form.

W_2 is a subspace of $M_{2,2}(\mathbb{R})$

- $\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix}$ for $a=b=0$. So, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W_2$.
- Let $x, y \in W_2$. Then $x = \begin{pmatrix} 0 & a_1 \\ -a_1 & b_1 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & a_2 \\ -a_2 & b_2 \end{pmatrix}$ where $a_1, b_1 \in \mathbb{R}$. Then $x+y = \begin{pmatrix} 0 & a_1+a_2 \\ -(a_1+a_2) & b_1+b_2 \end{pmatrix}$ is in W_2 since it is of the required form $\begin{pmatrix} 0 & a \\ -a & b \end{pmatrix}$.
- Let $x = \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in W_1$ and $\alpha \in \mathbb{R}$. Then $\alpha x = \begin{pmatrix} 0 & \alpha a \\ -(\alpha a) & \alpha b \end{pmatrix}$ is in W_2.

dim(W_1)

(b) Let $\begin{pmatrix} a & b \\ c & a \end{pmatrix} \in W_1$, where $a, b, c \in \mathbb{R}$.

Then

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

So, $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ spans W_1.

β is a linearly independent set since if

$$\alpha_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

then $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

And so, $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Thus, β is a basis for W_1.

So, $\dim(W_1) = 3$.

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dim(W_2)

Let $\begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in W_2$, where $a, b \in \mathbb{R}$.

Then $\begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} = a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

So, $\beta = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ spans W_2.

If $\alpha_1\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \alpha_2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ then $\begin{pmatrix} 0 & \alpha_1 \\ -\alpha_1 & \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Hence, $\alpha_1 = \alpha_2 = 0$. So, β is a linearly independent set.

So, β is a basis for W_2.

Thus, $\dim(W_2) = 2$.

$$\boxed{\dim(W_1 \cap W_2)}$$

$W_1 \cap W_2 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \middle| a,b,c \in \mathbb{R} \right\} \cap \left\{ \begin{pmatrix} 0 & \alpha \\ -\alpha & \beta \end{pmatrix} \middle| \alpha,\beta \in \mathbb{R} \right\}.$

If $X \in W_1 \cap W_2$, then $X = \begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ -\alpha & \beta \end{pmatrix}$ for some $a,b,c,\alpha,\beta \in \mathbb{R}$.

Then, $a=0$, $b=\alpha$, $c=-\alpha$, $a=\beta$.

So,

$$X = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}.$$

Then, $X = \alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$

So, $\beta = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ spans $W_1 \cap W_2$.

Since β consists of one non-zero vector,
by another HW problem in this HW set,
β is linearly independent.

Or just show it. If $c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $\begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and so $c=0$.

Thus, β is a basis for $W_1 \cap W_2$.

So, $\dim(W_1 \cap W_2) = 1$.

$\underline{\dim(W_1+W_2)=4}$ Method 1

We will show that $M_{2,2}(\mathbb{R}) = W_1+W_2$.

Let $\begin{pmatrix} x & y \\ w & z \end{pmatrix} \in M_{2,2}(\mathbb{R})$.

Let's try to solve for a, b, c, α, β in the following equation:

$$\begin{pmatrix} x & y \\ w & z \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & a \end{pmatrix}}_{\text{in } W_1} + \underbrace{\begin{pmatrix} 0 & \alpha \\ -\alpha & \beta \end{pmatrix}}_{\text{in } W_2}$$

This leads to trying to solve

$$\begin{aligned} a &= x \\ b+\alpha &= y \\ c-\alpha &= w \\ a+\beta &= z \end{aligned}$$

We can set $a=x, \beta=z-x, \alpha=0, b=y, c=w$.

So,

$$\begin{pmatrix} x & y \\ w & z \end{pmatrix} = \begin{pmatrix} x & y \\ w & x \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & z-x \end{pmatrix} \in W_1+W_2.$$

Hence $M_{2,2}(\mathbb{R}) \subseteq W_1+W_2$.

Since $W_1, W_2 \subseteq M_{2,2}(\mathbb{R})$ we know that $W_1+W_2 \subseteq M_{2,2}(\mathbb{R})$.

So, $M_{2,2}(\mathbb{R}) = W_1+W_2$.

$\underline{\dim(W_1+W_2)=4}$

$\boxed{\text{Method 2}}$

Let $X \in W_1+W_2$.

Then $X = w_1 + w_2$ where $w_1 = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$, $w_2 = \begin{pmatrix} 0 & \alpha \\ -\alpha & \beta \end{pmatrix}$

for some $a, b, c, \alpha, \beta \in \mathbb{R}$.

Note that

$$X = \begin{pmatrix} a & b \\ c & a \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ -\alpha & \beta \end{pmatrix}$$

$$(*) \quad = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \beta\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ span W_1+W_2.

However these vectors aren't linearly independent (there are 5 of them and we are in a 4 dimensional vector space $M_{2,2}(\mathbb{R})$).

For example, the vector $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\text{linear combo of } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$

So, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is redundant.

We can regroup and get that $(*)$ simplifies to

$$X = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (b+\alpha)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (c-\alpha)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \beta\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

So, $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

Span $W_1 + W_2$.

You can check and show that β is a linearly independent set.

Hence β is a basis for $W_1 + W_2$.

Thus, $\dim(W_1 + W_2) = 4$.

② Suppose $v_i = \vec{0}$. To simplify the notation of our proof lets assume that $i=1$. The same idea works for general i. If this is the case then we have the equation

$$1 \cdot \vec{0} + 0 \cdot v_2 + 0 \cdot v_3 + \cdots + 0 \cdot v_n = \vec{0}$$

That is,

$$1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n = \vec{0}.$$

Since not all the coefficients are equal to zero, this gives us our linear dependence equation.

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③ We show that $\{1, x, x^2, \ldots, x^n\}$ is a basis which will show that $P_n(F)$ has dimension n.

Let $f = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in P_n(F)$.

Then we see that

$$f = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n,$$

That is $\{1, x, x^2, \ldots, x^n\}$ spans $P_n(F)$.

Now suppose there exist constants $c_0, c_1, c_2, \ldots, c_n \in F$ with

$$c_0 + c_1 \cdot x + c_2 x^2 + \cdots + c_n x^n = \vec{0} \quad (*)$$

Then $c_0 = c_1 = \dots = c_n = 0$.

> This is where we are using that $F = \mathbb{R}$ or $\mathbb{C}$. The polynomial on the left of (*) has degree n. If it was non-zero then it can have up to n possible roots. But the zero polynomial on the right has an infinite number of roots. So for them to equal we must have that the polynomial on the left is the zero polynomial.

So, $\{1, x, x^2, \dots, x^n\}$ is a basis for $P_n(F)$.

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(4) Suppose that $P(\mathbb{R})$ is finite-dimensional with basis $f_1, f_2, \dots, f_n$ where each f_i is a polynomial with coefficients from $\mathbb{R}$. Suppose that d_i is the degree of f_i, ie the highest power of x that appears in f_i. Then if d is the maximum of all $d_1, d_2, \dots, d_n$ then x^{d+1} is not in the span of $f_1, f_2, \dots, f_n$. So, $f_1, f_2, \dots, f_n$ does not span $P(\mathbb{R})$. So, $P(\mathbb{R})$ is not finite dimensional.

(5) Let $x, y \in V$.

($\Rightarrow$) Suppose that $\{x, y\}$ is a linearly dependent set. Then there exist $c_1, c_2 \in F$ not both zero with $c_1 x + c_2 y = \vec{0}$.

If $c_1 \neq 0$, then $x = -c_1^{-1} c_2 y$.

If $c_2 \neq 0$, then $y = -c_2^{-1} c_1 x$.

So either x is a multiple of y or y is a multiple of x.

($\Leftarrow$)

Suppose that x is a multiple of y.

Then $x = \alpha y$ where $\alpha \in F$.

Then $1 \cdot x + (-\alpha) \cdot y = \vec{0}$.

not zero

So we have a linear dependence relation between x and y, that is, $\{x, y\}$ is a linearly dependent set.

The same proof works if y is a multiple of x.

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(6) Let $x \in V$ with $x \neq \vec{0}$. Suppose $cx = \vec{0}$ for some $c \in F$. If $c \neq 0$, then $c^{-1} c x = c^{-1} \vec{0}$ and so $x = \vec{0}$ which isn't true. Hence $c = 0$.

So, $\{x\}$ is a linearly independent set.

7 (a) Let $S = \{v_1, v_2, \dots, v_n\}$ be a linearly independent set of vectors and $v \in V$ with $v \notin S$.

($\Rightarrow$) Suppose that $S \cup \{v\}$ is linearly dependent.

Then there exist $c_1, c_2, \dots, c_n, c \in F$, not all zero, such that $c_1 v_1 + c_2 v_2 + \dots + c_n v_n + cv = \vec{0}$.

If $c = 0$, then the equation $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \vec{0}$ contradicts the fact that S is a linearly independent set since not all the c_i are zero.

Hence $c \neq 0$.

Thus, $v = -c^{-1}c_1 v_1 - c^{-1}c_2 v_2 - \dots - c^{-1}c_n v_n$.

So, $v \in \operatorname{span}(S)$.

($\Leftarrow$) If $v \in \operatorname{span}(S)$, then

$$v = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$$

for some $\beta_1, \beta_2, \dots, \beta_n \in F$.

So, $1 \cdot v - \beta_1 v_1 - \beta_2 v_2 - \dots - \beta_n v_n = \vec{0}$.

Since the coefficients in the above are not all zero, we get that $S \cup \{v\} = \{v_1, v_2, \dots, v_n, v\}$ is a linearly dependent set.

(b) Suppose that V is generated by some finite set S.

Here we assume that $V \neq \{\vec{0}\}$.

Thus, S contains some non-zero vector v_1. By problem 6, $\{v_1\}$

is a linearly independent set. Continue, if one can, to add elements from S to this set until one reaches a point where $\beta = \{v_1, v_2, \ldots, v_k\}$ is a linearly independent set and adding any more elements from S to β will yield a linearly dependent set.

We can do this since S is finite.

Let's now ~~show that β spans V.~~ We first show that $S \subseteq \text{span}(\beta)$.

Note that $v_i \in \text{Span}(\beta)$ since $v_i = 0v_1 + \cdots + 1v_i + \cdots + 0v_n$. Also, If $v \in S$ and $v \notin \beta$, then by the construction of β, we have that $\beta \cup \{v\}$ is linearly dependent and hence by part (a), $v \in \text{span}(\beta)$.

Thus, $S \subseteq \text{span}(\beta)$.

Now let $x \in V$.

Let us denote the elements of S by $S = \{\underbrace{v_1, v_2, \ldots, v_k}_{\beta}, \underbrace{v_{k+1}, \ldots, v_n}_{\text{not in } \beta}\}$.

Since $S \subseteq \text{span}(\beta)$ we have that

$$v_i = \sum_{j=1}^{k} a_{ij} v_j \quad \text{for } k+1 \leq i \leq n.$$

Now since $x \in V$ and V is spanned by S we have that

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k + \alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n$$
$$= \sum_{i=1}^{k} \alpha_i v_i + \sum_{i=k+1}^{n} \alpha_i \left(\sum_{j=1}^{k} a_{ij} v_j \right)$$
$$= \sum_{i=1}^{k} \alpha_i v_i + \sum_{j=1}^{k} \left(\sum_{i=k+1}^{n} \alpha_i a_{ij} \right) v_j$$

So, $x \in \text{span}(\beta)$.
Hence β is a linearly independent set that spans V.
So, β is a basis for V.

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8 (a) We know that $W_1 \cap W_2$ is a subspace of V by HW 1. Thus, since $W_1 \cap W_2 \subseteq W_1$, by the thm on pg 28 of the notes,
$\dim(W_1 \cap W_2) \leq \dim(W_1) = n$.

(b) If $W_1 = \{\vec{0}\}$, then $W_1 + W_2 = W_2$ and
$\dim(W_1 + W_2) = \dim(W_2) \leq m+n$.
If $W_2 = \{\vec{0}\}$, then $W_1 + W_2 = W_1$ and
$\dim(W_1 + W_2) = \dim(W_1) \leq m+n$.
So assume $W_1 \neq \{\vec{0}\}$ and $W_2 \neq \{\vec{0}\}$. Thus, W_1 and W_2 have bases $\beta_1 = \{w_1, \dots, w_m\}$ and $\beta_2 = \{v_1, \dots, v_n\}$.
Let $x \in W_1 + W_2$. Then $x = a + b$ where $a \in W_1$ and $b \in W_2$. So, $x = \underbrace{(\alpha_1 w_1 + \dots + \alpha_m w_m)}_{a} + \underbrace{(\gamma_1 v_1 + \dots + \gamma_n v_n)}_{b}$ for some $\alpha_i, \gamma_i \in F$.

So, $\beta_1 \cup \beta_2$ spans $W_1 + W_2$.
Note that the size of $\beta_1 \cup \beta_2$ is $\leq m+n$.

Since $W_1 \neq \{\vec{0}\}$ and $W_2 \neq \{\vec{0}\}$ we have that $W_1 + W_2 \neq \{\vec{0}\}$ (since W_1 and W_2 are contained in $W_1 + W_2$)

So, by #7, Some subset of $\beta_1 \cup \beta_2$ is a basis for $W_1 + W_2$.
Hence $\dim(W_1 + W_2) \leq m+n$.

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(9) Let $\beta = \{w_1, w_2, \ldots, w_k\}$ be a basis for W. If β spans V, then β is a basis for V. Otherwise there exists $v_{k+1} \in V$ with $v_{k+1} \notin \text{span}(\beta)$. By #7, $\beta \cup \{v_{k+1}\}$ is a linearly independent set. Now, if $\beta \cup \{v_{k+1}\}$ spans V, then $\beta \cup \{v_{k+1}\}$ is a basis for V. Otherwise there exists $v_{k+2} \in V$ with $v_{k+2} \notin \text{span}(\beta \cup \{v_{k+1}\})$. By #7, $\beta \cup \{v_{k+1}, v_{k+2}\}$ is linearly independent. Continue in this way until one reaches a linearly independant set $\beta' = \{w_1, \ldots, w_k, v_{k+1}, \ldots, v_n\}$ which spans V. This process must terminate since V is finite-dimensional of dimension n. That is,

$n+1$ or more vectors in V are linearly dependent. So, the process must stop once β' has n linearly independent vectors in it.