Homework #4

①

(a) $T(1,0,0) = (1,0,-1) = 1\cdot(1,0,0) + 0(0,1,0) + (-1)(0,0,1)$

$T(0,1,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$

$T(0,0,1) = (0,1,0) = 0(1,0,0) + 1(0,1,0) + 0(0,0,1)$

So, $[T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$

(b) $T(1,0,1) = (1,1,-1) = \frac{1}{2}(1,0,1) + \frac{1}{2}(1,2,1) + (-2)(0,0,1)$

$T(1,2,1) = (3,1,-1) = \frac{5}{2}(1,0,1) + \frac{1}{2}(1,2,1) + (-4)(0,0,1)$

$T(0,0,1) = (0,1,0) = -\frac{1}{2}(1,0,1) + \frac{1}{2}(1,2,1) + 0(0,0,1)$

So $[T]_{\beta'} = \begin{pmatrix} 1/2 & 5/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ -2 & -4 & 0 \end{pmatrix}$

Ex calculation from above

$(1,1,-1) = a(1,0,1) + b(1,2,1) + c(0,0,1)$

$(1,1,-1) = (a+b, 2b, a+b+c)$

$\left.\begin{aligned} a+b &= 1 \\ 2b &= 1 \\ a+b+c &= -1 \end{aligned}\right\} \begin{aligned} b &= 1/2 \\ a &= 1/2 \\ c &= -2 \end{aligned}$

other parts are similar

(c) $T(1,0,1) = (1,1,-1) = 1(1,0,0) + 1(0,1,0) - 1(0,0,1)$

$T(1,2,1) = (3,1,-1) = 3(1,0,0) + 1(0,1,0) - 1(0,0,1)$

$T(0,0,1) = (0,1,0) = 0(1,0,0) + 1(0,1,0) + 0(0,0,1)$

So, $[T]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix}$

(d) $x=(1,-2,4)$

$x=(1,-2,4)=1\cdot(1,0,0)-2(0,1,0)+4(0,0,1)$

So, $[(1,-2,4)]_\beta=(1,-2,4)$

$x=(1,-2,4)=a(1,0,1)+b(1,2,1)+c(0,0,1)$
$=(a+b,\ 2b,\ a+b+c)$

Solve; $\left.\begin{aligned} a+b &= 1 \\ 2b &= -2 \\ a+b+c &= 4 \end{aligned}\right\}\ \begin{aligned} &b=-1 \\ &a=2 \\ &c=4-a-b=4-2+1=3 \end{aligned}$

So, $x=2(1,0,1)-1(1,2,1)+3(0,0,1)$

$[(1,-2,4)]_{\beta'}=(2,-1,3).$

(e) $T(x)=T(1,-2,4)=(-1,4,-1)$
$=-1(1,0,0)+4(0,1,0)-1(0,0,1)$

So, $[T(x)]_\beta=(-1,4,-1)$

$[T]_\beta[x]_\beta=\begin{pmatrix}1&1&0\\0&0&1\\-1&0&0\end{pmatrix}\begin{pmatrix}1\\-2\\4\end{pmatrix}=\begin{pmatrix}1-2\\4\\-1\end{pmatrix}=\begin{pmatrix}-1\\4\\-1\end{pmatrix}$

(f) $T(x)=(-1,4,-1)=a(1,0,1)+b(1,2,1)+c(0,0,1)$
$=(a+b,\ 2b,\ a+b+c)$

$\left.\begin{aligned} a+b &= -1 \\ 2b &= 4 \\ a+b+c &= -1 \end{aligned}\right\}\ \begin{aligned} &b=2 \\ &a=-3 \\ &c=-1-a-b=-1+3-2=0 \end{aligned}$

So, $[T(x)]_{\beta'} = (-3, 2, 0)$

And
$$[T]_{\beta'}[x]_{\beta'} = \begin{pmatrix} 1/2 & 5/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ -2 & -4 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$$

So, $[T(x)]_{\beta'} = [T]_{\beta'}[x]_{\beta'}$.

(g) $[T(x)]_{\beta} = (-1, 4, -1)$ from part (e),

And $$[T]_{\beta'}^{\beta}[x]_{\beta'} = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$$

So, $[T(x)]_{\beta} = [T]_{\beta'}^{\beta}[x]_{\beta'}$

(h) $[I]_{\beta'}^{\beta}$ is calculated as follows

$I(1,0,1) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$
$I(1,2,1) = (1,2,1) = 1(1,0,0) + 2(0,1,0) + 1(0,0,1)$
$I(0,0,1) = (0,0,1) = 0(1,0,0) + 0(0,1,0) + 1(0,0,1)$

So, $$[I]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

(i) $[I]^{\beta}_{\beta'}[x]_{\beta'} = \begin{pmatrix}1&1&0\\0&2&0\\1&1&1\end{pmatrix}\begin{pmatrix}2\\-1\\3\end{pmatrix} = \begin{pmatrix}2-1\\-2\\2-1+3\end{pmatrix}$

$$= \begin{pmatrix}1\\-2\\4\end{pmatrix} = [x]_{\beta}$$

(j) We now calculate $[I]^{\beta'}_{\beta}$.

$I(1,0,0) = (1,0,0) = 1(1,0,1) + 0(1,2,1) - 1\cdot(0,0,1)$
$I(0,1,0) = (0,1,0) = -\frac{1}{2}(1,0,1) + \frac{1}{2}(1,2,1) + 0\,(0,0,1)$
$I(0,0,1) = (0,0,1) = 0(1,0,1) + 0(1,2,1) + 1(0,0,1)$

So, $[I]^{\beta'}_{\beta} = \begin{pmatrix}1&-1/2&0\\0&1/2&0\\-1&0&1\end{pmatrix}$

(k) $[I]^{\beta'}_{\beta}[x]_{\beta} = \begin{pmatrix}1&-1/2&0\\0&1/2&0\\-1&0&1\end{pmatrix}\begin{pmatrix}1\\-2\\4\end{pmatrix} = \begin{pmatrix}1+1\\-1\\-1+4\end{pmatrix}$

$$= \begin{pmatrix}2\\-1\\3\end{pmatrix} = [x]_{\beta'}$$

(l) $[I]^{\beta}_{\beta'}[I]^{\beta'}_{\beta} = \begin{pmatrix}1&1&0\\0&2&0\\1&1&1\end{pmatrix}\begin{pmatrix}1&-1/2&0\\0&1/2&0\\-1&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

and $[I]^{\beta'}_{\beta}[I]^{\beta}_{\beta'} = \begin{pmatrix}1&-1/2&0\\0&1/2&0\\-1&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&2&0\\1&1&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

So, $\left([I]^{\beta}_{\beta'}\right)^{-1} = [I]^{\beta'}_{\beta}$

(m)

$$[T]^{\beta}_{\beta'}[I]^{\beta'}_{\beta} = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$= [T]_{\beta}$$

(n)

$$[I]^{\beta'}_{\beta}[T]_{\beta}[I]^{\beta}_{\beta'} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 5/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ -2 & -4 & 0 \end{pmatrix}$$

$$= [T]_{\beta'}$$

(2) Let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$ be bases for V and W respectively,

(a) Let

$$T(v_1) = t_{11}w_1 + t_{21}w_2 + \cdots + t_{m1}w_m$$
$$T(v_2) = t_{12}w_1 + t_{22}w_2 + \cdots + t_{m2}w_m$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$T(v_n) = t_{1n}w_1 + t_{2n}w_2 + \cdots + t_{mn}w_m$$

and

$$S(v_1) = s_{11}w_1 + s_{21}w_2 + \cdots + s_{m1}w_m$$
$$S(v_2) = s_{12}w_1 + s_{22}w_2 + \cdots + s_{m2}w_m$$
$$\vdots \qquad\qquad \vdots$$
$$S(v_n) = s_{1n}w_1 + s_{2n}w_2 + \cdots + s_{mn}w_m$$

Thus,

$$(T+S)(v_1) = T(v_1) + S(v_1) = (t_{11}+s_{11})w_1 + (t_{21}+s_{21})w_2 + \cdots + (t_{m1}+s_{m1})w_m$$
$$(T+S)(v_2) = T(v_2) + S(v_2) = (t_{12}+s_{12})w_1 + (t_{22}+s_{22})w_2 + \cdots + (t_{m2}+s_{m2})w_m$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$(T+S)(v_n) = T(v_n) + S(v_n) = (t_{1n}+s_{1n})w_1 + (t_{2n}+s_{2n})w_2 + \cdots + (t_{mn}+s_{mn})w_m$$

So,

$$[T]_\beta^\gamma + [S]_\beta^\gamma = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{pmatrix} + \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & & \vdots \\ s_{m1} & s_{m2} & \cdots & s_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} t_{11}+s_{11} & t_{12}+s_{12} & \cdots & t_{1n}+s_{1n} \\ t_{21}+s_{21} & t_{22}+s_{22} & \cdots & t_{2n}+s_{2n} \\ \vdots & \vdots & & \vdots \\ t_{m1}+s_{m1} & t_{m2}+s_{m2} & \cdots & t_{mn}+s_{mn} \end{pmatrix} = [T+S]_\beta^\gamma$$

(b) Let $T(v_i) = \sum_{j=1}^{m} t_{ji} w_j$ be as in part a.

Since T is linear,

$$T(\alpha v_i) = \alpha T(v_i) = \sum_{j=1}^{m} (\alpha t_{ji}) w_j \text{ . So, } [\alpha T]_\beta^\gamma = \begin{pmatrix} \alpha t_{11} & \cdots & \alpha t_{1n} \\ \alpha t_{21} & \cdots & \alpha t_{2n} \\ \vdots & \ddots & \vdots \\ \alpha t_{m1} & \cdots & \alpha t_{mn} \end{pmatrix}$$

So,

$$\alpha [T]_\beta^\gamma = \alpha \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{pmatrix} =$$

$$= \begin{pmatrix} \alpha t_{11} & \alpha t_{12} & \cdots & \alpha t_{1n} \\ \alpha t_{21} & \alpha t_{22} & \cdots & \alpha t_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha t_{m1} & \alpha t_{m2} & \cdots & \alpha t_{mn} \end{pmatrix} = [\alpha T]_\beta^\gamma$$

3

(a) Let $c_1, c_2 \in F$ and $v_1, v_2 \in V$. Then

$$(U \circ T)(c_1 v_1 + c_2 v_2) = U(T(c_1 v_1 + c_2 v_2)) =$$

$$\underset{\text{Since T is linear}}{=} U(c_1 T(v_1) + c_2 T(v_2)) \underset{\text{Since U is linear}}{=} c_1 U(T(v_1)) + c_2 U(T(v_2))$$

$$= c_1 (U \circ T)(v_1) + (U \circ T)(v_2)$$

So, $U \circ T$ is linear.

(b) Let $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]$, $\beta = [\beta_1, \beta_2, \ldots, \beta_m]$ and $\gamma = [\gamma_1, \gamma_2, \ldots, \gamma_p]$ be bases for V, W, and Z respectively.

Let

$$\begin{aligned}
T(\alpha_1) &= t_{11}\beta_1 + t_{21}\beta_2 + \cdots + t_{m1}\beta_m \\
T(\alpha_2) &= t_{12}\beta_1 + t_{22}\beta_2 + \cdots + t_{m2}\beta_m \\
&\vdots \\
T(\alpha_n) &= t_{1n} + t_{2n}\beta_2 + \cdots + t_{mn}\beta_m
\end{aligned}$$

and

$$\begin{aligned}
U(\beta_1) &= u_{11}\gamma_1 + u_{21}\gamma_2 + \cdots + u_{p1}\gamma_p \\
U(\beta_2) &= u_{12}\gamma_1 + u_{22}\gamma_2 + \cdots + u_{p2}\gamma_p \\
&\vdots \\
U(\beta_m) &= u_{1m}\gamma_1 + u_{2m}\gamma_2 + \cdots + u_{pm}\gamma_p
\end{aligned}$$

ie. $T(\alpha_i) = \sum_{j=1}^{m} t_{ji}\beta_j$ and $U(\beta_i) = \sum_{k=1}^{p} u_{ki}\gamma_k$

Then

$$(U \circ T)(\alpha_i) = U(T(\alpha_i)) = \underline{U\left(\sum_{j=1}^{m} t_{ji}\beta_j\right)}$$

$$= \sum_{j=1}^{m} t_{ji}\, U(\beta_j) = \sum_{j=1}^{m} t_{ji} \sum_{k=1}^{p} u_{kj}\gamma_k$$

$\boxed{\text{since } U \text{ is linear}}$

$$= \sum_{k=1}^{p} \left(\sum_{j=1}^{m} t_{ji} u_{kj} \right) \gamma_k$$

Note that

$$[T]_\alpha^\beta = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{pmatrix}$$ and

$$[U]_\beta^\gamma = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ u_{21} & u_{22} & \cdots & u_{2m} \\ \vdots & \vdots & & \vdots \\ u_{p1} & u_{p2} & \cdots & u_{pm} \end{pmatrix}.$$

So,

$$\underbrace{\underbrace{[U]_\beta^\gamma}_{p \times m} \underbrace{[T]_\alpha^\beta}_{m \times n}}_{p \times n} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{pmatrix}$$ where

$$c_{ki} = \sum_{j=1}^{m} u_{kj} t_{ji}$$

by the def. of mutrix multiplication.

And,

$$(U \circ T)(\alpha_i) = \sum_{k=1}^{p} \left[\sum_{j=1}^{m} t_{ji} u_{kj} \right] \gamma_k$$

$$= \sum_{k=1}^{p} c_{ki} \gamma_k .$$

Thus, $[U \circ T]_\alpha^\gamma = [U]_\beta^\gamma [T]_\gamma^\beta .$

(4) Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\beta = \{\beta_1, \beta_2, \ldots, \beta_m\}$ be bases for V and W respectively.

Suppose that

$$[T_1]_\alpha^\beta = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = [T_2]_\alpha^\beta$$

Then, by the def. of $[T_1]_\alpha^\beta$ and $[T_2]_\alpha^\beta$ we have

$$T_1(\alpha_1) = a_{11}\beta_1 + a_{21}\beta_2 + \cdots + a_{m1}\beta_m = T_2(\alpha_1)$$
$$T_1(\alpha_2) = a_{12}\beta_1 + a_{22}\beta_2 + \cdots + a_{m2}\beta_m = T_2(\alpha_2)$$
$$\vdots$$
$$T_1(\alpha_n) = a_{1n}\beta_1 + a_{22}\beta_2 + \cdots + a_{mn}\beta_m = T_2(\alpha_n)$$

Since T_1 and T_2 agree on all the vectors in α they agree on all of V. Why?

Let $x \in V$. Then $x = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n$ for some $b_i \in F$. So,

$$\begin{aligned} T_1(x) &= T_1(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n) \\ &= b_1T_1(\alpha_1) + b_2T_1(\alpha_2) + \cdots + b_nT_1(\alpha_n) \\ &= b_1T_2(\alpha_1) + b_2T_2(\alpha_2) + \cdots + b_nT_2(\alpha_n) \\ &= T_2(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n) = T_2(x). \end{aligned}$$

⑤ Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ and $L_A(x) = Ax$.

(a) Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ldots, v_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

Then, $L_A(v_1) = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} = a_{11}v_1 + a_{21}v_2 + \cdots + a_{n1}v_n$

$$L_A(v_2) = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} = a_{12}v_1 + a_{22}v_2 + \cdots + a_{n2}v_n$$

$$\vdots \qquad \qquad \vdots$$

$$L_A(v_n) = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix} = a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{nn}v_n.$$

So,

$$[L_A]_\beta = \left([L_A(v_1)]_\beta \,\middle|\, [L_A(v_2)]_\beta \,\middle|\, \cdots \,\middle|\, [L_A(v_n)]_\beta \right)$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = A$$

(b) From class, we know that L_A is invertible iff $[L_A]_\beta$ is invertible. By part a, we then have that L_A is invertible iff A is invertible.

(c) We know from class that

$$[L_A]_\gamma = [I]_\beta^\gamma [L_A]_\beta [I]_\gamma^\beta = \left([I]_\gamma^\beta\right)^{-1} [L_A]_\beta [I]_\gamma^\beta$$

Since $[L_A]_\beta = A$, we just need to show that $[I]_\gamma^\beta = (w_1 | w_2 | \cdots | w_n)$.

Let $w_1 = \begin{pmatrix} w_{11} \\ w_{21} \\ \vdots \\ w_{n1} \end{pmatrix}$, $w_2 = \begin{pmatrix} w_{12} \\ w_{22} \\ \vdots \\ w_{n2} \end{pmatrix}$, ..., $w_n = \begin{pmatrix} w_{1n} \\ w_{2n} \\ \vdots \\ w_{nn} \end{pmatrix}$.

Then,

$$\begin{aligned} w_1 &= w_{11} v_1 + w_{21} v_2 + \cdots + w_{n1} v_n \\ w_2 &= w_{12} v_1 + w_{22} v_2 + \cdots + w_{n2} v_n \\ &\vdots \\ w_n &= w_{1n} v_1 + w_{2n} v_2 + \cdots + w_{nn} v_n \end{aligned}$$

So, $$[I]_\gamma^\beta = \left([w_1]_\beta \middle| [w_2]_\beta \middle| \cdots \middle| [w_n]_\beta \right) = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{pmatrix} = (w_1 | w_2 | \cdots | w_n)$$