Homework 5

①

(a) Let $\gamma = \left[\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right]$.

$$\left.\begin{aligned} T\begin{pmatrix}1\\0\\0\end{pmatrix} &= \begin{pmatrix}4\\2\\1\end{pmatrix} \\ T\begin{pmatrix}0\\1\\0\end{pmatrix} &= \begin{pmatrix}0\\3\\0\end{pmatrix} \\ T\begin{pmatrix}0\\0\\1\end{pmatrix} &= \begin{pmatrix}1\\2\\4\end{pmatrix} \end{aligned}\right\} \quad [T]_\gamma = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$$

(i)

$$\det(T - \lambda I) = \det\left([T]_\gamma - \lambda I\right) = \det\begin{pmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{pmatrix}$$

expand on 1st row

$$= (4-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 0 & 4-\lambda \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 1 & 4-\lambda \end{vmatrix} + 1\cdot\begin{vmatrix} 2 & 3-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= (4-\lambda)(3-\lambda)(4-\lambda) + 0 + 1\cdot[-(3-\lambda)]$$

$$= (3-\lambda)[(4-\lambda)(4-\lambda) - 1]$$

$$= (3-\lambda)[15 - 8\lambda + \lambda^2] = (3-\lambda)(\lambda-5)(\lambda-3)$$

$$= -(\lambda-3)^2(\lambda-5)$$

The eigenvalues are 3 and 5.

(ii) Eigenspace for $\lambda = 3$:

Need to solve $T\begin{pmatrix}a\\b\\c\end{pmatrix} = 3\begin{pmatrix}a\\b\\c\end{pmatrix}$ or $\begin{pmatrix}4a+c\\2a+3b+2c\\a+4c\end{pmatrix} = \begin{pmatrix}3a\\3b\\3c\end{pmatrix}$

That is, $\begin{matrix} a+c=0 \\ 2a+2c=0 \\ a+c=0 \end{matrix}$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[-R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Big\} \; a+c=0.$$

So, $\begin{matrix} a=-t \\ b=s \\ c=t \end{matrix}$ where $t, s \in \mathbb{R}$.

Thus, $E_3(T) = \left\{ \begin{pmatrix} -t \\ s \\ t \end{pmatrix} \middle| \; t,s \in \mathbb{R} \right\} = \left\{ t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| \; t,s \in \mathbb{R} \right\}$

So, a basis is $\gamma_1 = \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$, (You can see they are linearly independent since they aren't multiples of each other)

so, $\lambda = 3$ has geometric mult. equal to 2.

Eigenspace for $\lambda = 5$:

We need to solve $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 5\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ or $\begin{pmatrix} 4a+c \\ 2a+3b+2c \\ a+4c \end{pmatrix} = \begin{pmatrix} 5a \\ 5b \\ 5c \end{pmatrix}$

That is, $\left.\begin{matrix} -a+c=0 \\ 2a-2b+2c=0 \\ a-c=0 \end{matrix}\right\}$ $\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 2 & -2 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \longrightarrow$

$$\xrightarrow[R_1+R_3 \to R_3]{2R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[-\frac{1}{2}R_2 \to R_2]{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{matrix} a-c=0 \\ b-2c=0 \end{matrix}$$

$\left.\begin{matrix} c=t \\ a=t \\ b=2t \end{matrix}\right\}$ So, $E_5(T) = \left\{ \begin{pmatrix} t \\ 2t \\ t \end{pmatrix} \middle| \; t \in \mathbb{R} \right\} = \left\{ t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \middle| \; t \in \mathbb{R} \right\}$

Let $\beta_2 = \left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right]$. Then β_2 is a basis for $E_5(T)$.

So, $\lambda = 3$ has geometric multiplicity 1.

Let $\beta = \beta_1 \cup \beta_2 = \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right]$.

Then β is a basis for $\mathbb{R}^3$ since β has 3 elements.

(iii)

eigenvalue	geometric multi	algebraic mult.
$\lambda = 3$	2	2
$\lambda = 5$	1	1

(iv) Yes T is diagonalizable since

$$[T]_\beta = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

(b)

(i) Let $\gamma = [1, x, x^2]$.

$T(1) = 1 + (x+1)\cdot 0 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$T(x) = x + (x+1)\cdot 1 = 1\cdot 1 + 2\cdot x + 0\cdot x^2$

$T(x^2) = x^2 + (x+1)\cdot 2x = 2x + 3x^2 = 0\cdot 1 + 2\cdot x + 3\cdot x^2$

So, $[T]_\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$.

So,

$$\det(T-\lambda I) = \det([T]_\gamma - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 0 & 3-\lambda \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 2-\lambda & 2 \end{vmatrix}$$

expand on 1st column

$$= (1-\lambda)(2-\lambda)(3-\lambda)$$

So, $\lambda = 1, 2, 3$ are the eigenvalues of T.

(ii) Let's calculate a basis for $E_1(T)$.

We need to solve $T(a+bx+cx^2) = a+bx+cx^2$.

That is, $(a+bx+cx^2)+(x+1)(b+2cx) = a+bx+cx^2$,

Ie, $a+bx+cx^2+bx+2cx^2+b+2cx-a-bx-cx^2=0$,

So, $b+(b+2c)x+2cx^2=0$

That is, $\left.\begin{array}{l} b=0 \\ b+2c=0 \\ 2c=0 \end{array}\right\}$ $\begin{array}{l} a=t \\ b=0 \\ c=0 \end{array}$.

So, $E_1(T) = \{t \cdot 1 \mid t \in \mathbb{R}\}$.

So, a basis is $\beta_1 = [1]$. So, $\lambda=1$ has geometric mult equal to 1,

Now let's calculate $E_2(T)$.

Need to solve $T(a+bx+cx^2) = 2(a+bx+cx^2)$

$(a+b)+(2b+2c)x+3cx^2 = 2a+2bx+2cx^2$

$(-a+b)+2cx+cx^2=0$.

Need to solve $\left.\begin{array}{l} -a+b=0 \\ 2c=0 \\ -c=0 \end{array}\right\}$ $\begin{array}{l} a=b=t \\ c=0 \end{array}$ $\longrightarrow$

So, $E_2(T) = \{ t + tx + 0x^2 \mid t \in \mathbb{R} \}$

$= \{ t(1+x) \mid t \in \mathbb{R} \}$

Let $\beta_2 = [1+x]$. β_2 is a basis for $E_2(T)$.

So, $\lambda = 2$ has geometric mult. equal to 1.

Now we calculate $E_3(T) = \{ a+bx+cx^2 \mid T(a+bx+cx^2) = 3(a+bx+cx^2) \}$

So, we need to solve

$$(a+b) + (2b+2c)x + 3cx^2 = 3(a+bx+cx^2)$$

That is, $(-2a+b) + (-b+2c)x = 0$

Need to solve

$$\left.\begin{aligned} -2a+b &= 0 \\ -b+2c &= 0 \end{aligned}\right\} \longrightarrow \boxed{\begin{aligned} a &= \tfrac{1}{2}b = t \\ b &= 2t \qquad t \in \mathbb{R} \\ c &= t \end{aligned}}$$

So, $E_3(T) = \{ t + 2tx + tx^2 \mid t \in \mathbb{R} \}$

$= \{ t(1+2x+x^2) \mid t \in \mathbb{R} \}$

So, $\beta_3 = [1+2x+x^2]$ is an ordered basis for $E_3(T)$

So, $\lambda = 3$ has geometric mult. equal to 1.

(iii)

eigenvalue	alg. mult.	geom. mult.
$\lambda = 1$	1	1
$\lambda = 2$	1	1
$\lambda = 3$	1	1

(iv) Yes. Let

$$\beta = \beta_1 \cup \beta_2 \cup \beta_3 = [1, 1+x, 1+2x+x^2].$$

Since β has 3 elements and $P_2(\mathbb{R})$ has dimension 3 we get a basis of eigenvectors for $P_2(\mathbb{R})$. Thus, T is diagonalizable.

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1 (c)

(i) Let $\gamma = \left[\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right]$ be the standard basis for $\mathbb{R}^3$.

Then

$$\left.\begin{aligned}
T\begin{pmatrix}1\\0\\0\end{pmatrix} &= \begin{pmatrix}3\\0\\0\end{pmatrix} = 3\begin{pmatrix}1\\0\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\0\\1\end{pmatrix}\\
T\begin{pmatrix}0\\1\\0\end{pmatrix} &= \begin{pmatrix}1\\3\\0\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\0\\1\end{pmatrix}\\
T\begin{pmatrix}0\\0\\1\end{pmatrix} &= \begin{pmatrix}0\\0\\4\end{pmatrix} = 0\begin{pmatrix}1\\0\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\0\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\end{pmatrix}
\end{aligned}\right\}$$

So,

$$[T]_\gamma = \begin{pmatrix}3 & 1 & 0\\ 0 & 3 & 0\\ 0 & 0 & 4\end{pmatrix}$$

So,

$$\det(T-\lambda I) = \det([T]_\gamma - \lambda I) = \det\begin{pmatrix}3-\lambda & 1 & 0\\ 0 & 3-\lambda & 0\\ 0 & 0 & 4-\lambda\end{pmatrix}$$

$$= (3-\lambda)\begin{vmatrix}3-\lambda & 0\\ 0 & 4-\lambda\end{vmatrix} - 0\begin{vmatrix}1 & 0\\ 0 & 4-\lambda\end{vmatrix} + 0\begin{vmatrix}1 & 0\\ 3-\lambda & 0\end{vmatrix}$$

expand on 1st column

$$= (3-\lambda)(3-\lambda)(4-\lambda)$$

So, the eigenvalues / algebraic mult.'s are $\lambda = 3$ w/ alg. mult. of 2 and $\lambda = 4$ with alg. mult of 1.

(ii)

$$E_3(T)=\left\{\begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; T\begin{pmatrix}a\\b\\c\end{pmatrix}=3\begin{pmatrix}a\\b\\c\end{pmatrix}\right\}$$

$$=\left\{\begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; \begin{pmatrix}3a+b\\3b\\4c\end{pmatrix}=\begin{pmatrix}3a\\3b\\3c\end{pmatrix}\right\}=\left\{\begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; \begin{pmatrix}b\\0\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$$

$$=\left\{\begin{pmatrix}a\\0\\0\end{pmatrix} \;\middle|\; a\in\mathbb{R}\right\}=\left\{t\begin{pmatrix}1\\0\\0\end{pmatrix} \;\middle|\; t\in\mathbb{R}\right\}$$

So, an ordered basis for $E_3(T)$ is $\beta_1=\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\right]$.

So, $\lambda=3$ has geometric multi. equal to 1.

$$E_4(T)=\left\{\begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; T\begin{pmatrix}a\\b\\c\end{pmatrix}=4\begin{pmatrix}a\\b\\c\end{pmatrix}\right\}=\left\{\begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; \begin{pmatrix}3a+b\\3b\\4c\end{pmatrix}=\begin{pmatrix}4a\\4b\\4c\end{pmatrix}\right\}$$

$$=\left\{\begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; \begin{pmatrix}-a+b\\-b\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$$

We get the equations

$$\left.\begin{aligned}-a+b&=0\\-b&=0\end{aligned}\right\}\quad \text{So } \begin{aligned}&a=0\\&b=0\\&c=\text{anything}\end{aligned}$$

Thus,

$$E_4(T)=\left\{\begin{pmatrix}0\\0\\c\end{pmatrix} \;\middle|\; c\in\mathbb{R}\right\}=\left\{t\begin{pmatrix}0\\0\\1\end{pmatrix} \;\middle|\; t\in\mathbb{R}\right\}$$

Thus, $\beta_2=\left[\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$ is an ordered basis for $E_4(T)$.

So $\lambda=4$ has geometric multi. equal to 1.

(iii)

eigenvalue	algebraic multiplicity	geometric multiplicity
$\lambda=3$	2	1
$\lambda=4$	1	1

(iv) Since $\beta=\beta_1\cup\beta_2=\left[\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$ has only 2 elements it doesn't span $\mathbb{R}^3$.
So we don't get a basis of eigenvectors.
So T is not diagonalizable.

1(d)

(i) Let $\gamma=\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right]$ be the standard basis for $\mathbb{C}^2$.

Then
$$\left.\begin{aligned}T\begin{pmatrix}1\\0\end{pmatrix}&=\begin{pmatrix}1\\i\end{pmatrix}=1\cdot\begin{pmatrix}1\\0\end{pmatrix}+i\begin{pmatrix}0\\1\end{pmatrix}\\ T\begin{pmatrix}0\\1\end{pmatrix}&=\begin{pmatrix}i\\1\end{pmatrix}=i\begin{pmatrix}1\\0\end{pmatrix}+1\cdot\begin{pmatrix}0\\1\end{pmatrix}\end{aligned}\right\}\quad \text{So, } [T]_\gamma=\begin{pmatrix}1&i\\i&1\end{pmatrix}$$

Then,
$$\det(T-\lambda I)=\det\left(\begin{pmatrix}1&i\\i&1\end{pmatrix}-\lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)=\det\begin{pmatrix}1-\lambda&i\\i&1-\lambda\end{pmatrix}$$
$$=(1-\lambda)(1-\lambda)-i\cdot i=1-2\lambda+\lambda^2+1=\lambda^2-2\lambda+2$$

The eigenvalues are

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm \sqrt{4} \cdot i}{2} = 1 \pm i .$$

$$\boxed{\begin{array}{c} \sqrt{-a} = \sqrt{a}\, i \\ a > 0 \end{array}}$$

So, $\lambda = 1+i$ has alg. mult. equal to 1, and $\lambda = 1-i$ has alg. mult equal to 1,

(ii)

$$E_{1+i}(T) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \,\middle|\, T\begin{pmatrix} x \\ y \end{pmatrix} = (1+i)\begin{pmatrix} x \\ y \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \,\middle|\, \begin{pmatrix} x+iy \\ ix+y \end{pmatrix} = \begin{pmatrix} x+ix \\ y+iy \end{pmatrix} \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \,\middle|\, \begin{pmatrix} iy-ix \\ ix-iy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

So we want to solve $\boxed{\begin{array}{l} -ix + iy = 0 \\ ix - iy = 0 \end{array}}$

$$\left(\begin{array}{cc|c} -i & i & 0 \\ i & -i & 0 \end{array}\right) \xrightarrow{R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} -i & i & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{iR_1 \to R_1} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

This system is now

$$\left.\begin{array}{r} x - y = 0 \\ 0 = 0 \end{array}\right\} \longrightarrow \boxed{\begin{array}{l} x = y = t \\ y = t \\ t \in \mathbb{C} \end{array}}$$

So, $E_{1+i}(T) = \left\{ \begin{pmatrix} t \\ t \end{pmatrix} \,\middle|\, t \in \mathbb{C} \right\} = \left\{ t\begin{pmatrix} 1 \\ 1 \end{pmatrix} \,\middle|\, t \in \mathbb{C} \right\}$

Thus, $\beta_1 = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$ is a basis for $E_{1+i}(T)$. So, $\dim(E_{1+i}(T)) = 1$

$$E_{1-i}(T) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \,\middle|\, T\begin{pmatrix} x \\ y \end{pmatrix} = (1-i)\begin{pmatrix} x \\ y \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \,\middle|\, \begin{pmatrix} x+iy \\ ix+y \end{pmatrix} = \begin{pmatrix} x-ix \\ y-iy \end{pmatrix} \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \,\middle|\, \begin{pmatrix} ix+iy \\ ix+iy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

So, the system to solve is $\boxed{\begin{matrix} ix+iy=0 \\ ix+iy=0 \end{matrix}} \longleftrightarrow \boxed{\begin{matrix} x+y=0 \\ x+y=0 \end{matrix}} \Updownarrow \boxed{\begin{matrix} x=-y=-t \\ y=t \\ t\in\mathbb{C} \end{matrix}}$

Thus,

$$E_{1-i}(T) = \left\{ \begin{pmatrix} -t \\ t \end{pmatrix} \,\middle|\, t\in\mathbb{C} \right\}$$
$$= \left\{ t\begin{pmatrix} -1 \\ 1 \end{pmatrix} \,\middle|\, t\in\mathbb{C} \right\}$$

So, $\beta_2 = \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]$ is a basis for $E_{1-i}(T)$.

So, $\dim(E_{1-i}(T)) = 1$.

(iii)

eigenvalue	geometric mult	algebraic mult
$\lambda = 1+i$	1	1
$\lambda = 1-i$	1	1

(iv) T is diagonalizable since $\beta = \beta_1 \cup \beta_2 = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]$ is a basis of eigenvectors for $\mathbb{C}^2$.

① (e) Let $T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be given by $T(f) = f' + f''$

(i) Let $\gamma = [1, x, x^2, x^3]$ be a basis for $P_3(\mathbb{R})$.

Then

$$T(1) = 0 + 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3$$
$$T(x) = 1 + 0 = 1\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3$$
$$T(x^2) = 2x + 2 = 2\cdot 1 + 2\cdot x + 0\cdot x^2 + 0\cdot x^3$$
$$T(x^3) = 3x^2 + 6x = 0\cdot 1 + 6\cdot x + 3\cdot x^2 + 0\cdot x^3$$

So, $[T]_\gamma = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

And

$$\det(T - \lambda I) = \det \begin{pmatrix} -\lambda & 1 & 2 & 0 \\ 0 & -\lambda & 2 & 6 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = \lambda^4$$

So the only eigenvalue is $\lambda = 0$.

(ii) $E_0(T) = \{a + bx + cx^2 + dx^3 \mid T(a + bx + cx^2 + dx^3) = 0\}$

$= \{a + bx + cx^2 + dx^3 \mid (b + 2cx + 3dx^2) + (2c + 6dx) = 0\}$

$= \{a + bx + cx^2 + dx^3 \mid (b + 2c) + (2c + 6d)x + 3dx^2 = 0\}$

So we need to solve

$$\boxed{\begin{aligned} b + 2c &= 0 \\ 2c + 6d &= 0 \\ 3d &= 0 \end{aligned}} \longrightarrow \boxed{\begin{aligned} d &= 0 \\ 2c &= -6d = 0 \\ b &= -2c = 0 \\ a &= t \end{aligned}}$$

So, $E_0(T) = \{t \mid t \in \mathbb{R}\} = \{t \cdot 1 \mid t \in \mathbb{R}\}$

Thus, $\beta = [1]$ is a basis for E_0.

(iii)

eigenvalue	geometric mult	algebraic mult
$\lambda = 0$	1	4

(iv) $\beta = [1]$ is not a basis for $P_3(\mathbb{R})$.
So, T is not diagonalizable.

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(2) We have that

$$I(v_1) = v_1 = 1 \cdot v_1 + 0v_2 + 0v_3 + \cdots + 0v_n$$
$$I(v_2) = v_2 = 0 \cdot v_1 + 1 \cdot v_2 + 0v_3 + \cdots + 0v_n$$
$$I(v_3) = v_3 = 0v_1 + 0v_2 + 1v_3 + \cdots + 0v_n$$
$$\vdots \qquad\qquad \vdots$$
$$I(v_n) = v_n = 0v_1 + 0v_2 + 0v_3 + \cdots + 1v_n$$

So, $$[I]_\beta = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = I_{n \times n}$$

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(3) (a) Let β be an ordered basis for V.

($\Rightarrow$) Suppose that T is diagonalizable. Then there exists an ordered basis of eigenvectors of T, call it γ, where $[T]_\gamma$ is a diagonal matrix. Let $Q = [I]_\gamma^\beta$ be the change of basis matrix from γ to β.

Then $Q^{-1}[T]_\beta Q = [T]_\gamma$.

So, $[T]_\beta$ is diagonalizable.

($\Leftarrow$) Suppose that $[T]_\beta$ is diagonalizable.
That is, there exists an invertible $n \times n$ matrix Q such that $Q^{-1}[T]_\beta Q = D$ where D is a diagonal matrix.
Thus, D is a matrix that is similar to $[T]_\beta$.

From class notes we saw that this implies that there exists a basis γ for V where $[T]_\gamma = D$.

Hence T is diagonalizable.

(3b) Let β be the standard basis for F^n.
Then by part (a), L_A is diagonalizable
iff $[L_A]_\beta$ is diagonalizable
iff A is diagonalizable.
[Here we are using the fact that $[L_A]_\beta = A$ when β is the standard basis for F^n. This was proved in HW 4.]

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(4) We have that $\left([I]_\gamma^\beta\right)^{-1}[T]_\beta [I]_\gamma^\beta = [T]_\gamma$.

So,

$$\det([T]_\gamma) = \det\left(\left([I]_\gamma^\beta\right)^{-1}\right)\det([T]_\beta)\det\left([I]_\gamma^\beta\right)$$

$$= \frac{1}{\det\left([I]_\gamma^\beta\right)}\det([T]_\beta)\det\left([I]_\gamma^\beta\right)$$

$$= \det([T]_\beta).$$ $\boxtimes$

$\det(ABC) = \det(A)\det(B)\det(C)$

$\det(A^{-1}) = \dfrac{1}{\det(A)}$

5

(a) From class, T is invertible iff $[T]_\beta$ is invertible. Thus T is invertible iff $[T]_\beta$ is invertible iff $\det([T]_\beta) \neq 0$ iff $\det(T) \neq 0$.

(b) If T is invertible then $\det(T) = \det([T]_\beta) \neq 0$. So,

$$\det(T^{-1}) = \det([T^{-1}]_\beta) \underset{\boxed{\text{Thm in class}}}{=} \det([T]_\beta^{-1})$$

$$\underset{\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}}{=} \frac{1}{\det([T]_\beta)} = \frac{1}{\det(T)}$$

(c) $\det(T \circ S) = \det([T \circ S]_\beta)$

$$\underset{\boxed{\text{HW } 4}}{=} \det([T]_\beta [S]_\beta) \underset{\boxed{\det(AB) = \det(A)\det(B)}}{=} \det([T]_\beta)\det([S]_\beta)$$

$$= \det(T)\det(S).$$

⑥

(a) • We have that $T(\vec{0}) = \vec{0} = \lambda\vec{0}$.
So, $\vec{0} \in E_\lambda(T)$.

• Let $x, y \in E_\lambda(T)$.
Then $T(x) = \lambda x$ and $T(y) = \lambda y$.
So, $T(x+y) = T(x) + T(y) = \lambda x + \lambda y = \lambda(x+y)$.
So, $x+y \in E_\lambda(T)$.

• Let $\alpha \in F$ and $x \in E_\lambda(T)$.
So, $T(x) = \lambda x$.
Then $T(\alpha x) = \alpha T(x) = \alpha\lambda x = \lambda(\alpha x)$.
So, $\alpha x \in E_\lambda(T)$.

Thus from above, $E_\lambda(T)$ is a subspace of V.

(b) Let $v_1, v_2, \ldots, v_r \in E_\lambda(T)$.
Then $T(v_i) = \lambda v_i$ for $i = 1, \ldots, r$.

$$
\begin{aligned}
\text{So, } & T(c_1v_1 + c_2v_2 + \cdots + c_rv_r) \\
&= c_1T(v_1) + c_2T(v_2) + \cdots + c_rT(v_r) \\
&= c_1\lambda v_1 + c_2\lambda v_2 + \cdots + c_r\lambda v_r \\
&= \lambda(c_1v_1 + c_2v_2 + \cdots + c_rv_r).
\end{aligned}
$$

Thus, $c_1v_1 + c_2v_2 + \cdots + c_rv_r \in E_\lambda(T)$.