Spring 2020 Math 4570
Dr. D. Liu Final Exam (May 11) Name

▷ ▷ SHOW ALL YOUR WORK AND BOX FINAL ANSWERS!

1. (5 pts) Solve the following system. Express the solution as the sum of a particular solution and solutions of the homogeneous system.

$$\begin{cases} x & & -z & & = & 1 \\ & y & +2z & -w & = & 3 \\ x & +2y & +3z & -w & = & 7. \end{cases}$$

2. (5 pts) Prove or disprove: All the real-valued functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(2) = 0$ form a vector space.

3. (5 pts) Determine whether the following set forms a basis for P_2: $\{2 + x + 7x^2, 3 - x + 2x^2, 5 - 3x^2\}$. Justify your reasons.

4. (5 pts) Show that the following subset of $\mathbb{R}^2$ is linearly independent if and only $ad - bc \neq 0$.

$$\left\{ \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right\}.$$

5. (7 pts) (a) State the definition of a basis for a vector space V. (b) Let $B = \langle b_1, b_2, \ldots, b_n \rangle$ be a basis for a vector space V. Prove: $B^* = \langle b_1 + b_2, b_2, b_3, \ldots, b_n \rangle$ is also a basis for V (without using the exchange lemma directly).

6. (5 pts) Prove or disprove: Let A be an $n \times m$ matrix where $n \neq m$. Then the row space of A is isomorphic to its column space.

7. (7 pts) Let f be the derivative function from $\mathcal{P}^3$ to $\mathcal{P}^3$.

   (a) Prove f is a homomorphism.

   (b) Find the null space and the range space of f.

   (c) Is f one-one? Is f onto?

   (d) Find the representation matrix of f with respect to the standard basis.

8. (10 pts) Let $h : \mathcal{P}_2 \to \mathbb{M}_{2\times 2}$ be defined by

$$a_0 + a_1 x + a_2 x^2 \mapsto \begin{pmatrix} a_0 + a_1 & a_1 + 2a_2 - a_0 \\ 3a_2 + 2a_1 & a_0 - a_2 \end{pmatrix}.$$

   Let $B = \langle x^2, x + 1, 1 \rangle$ and

$$D = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle;$$

   Determine the following

   (a) The representing matrix for h with respect to B and D.

   (b) Is the following in the range?

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

   (c) Is h one-one? Is h onto? Why or why not?

9. (9 pts) For the following matrix A, (a) find Adj(A) (the adjoint matrix of A); (b) use any method to find the determinant of A; (c) find $A\times$Adj(A); (d) find A^{-1}, the inverse of A.

$$A = \begin{pmatrix} 3 & 2 & 3 \\ 1 & 3 & -2 \\ 2 & 8 & 3 \end{pmatrix}.$$

10. (7 pts) Suppose the matrix A in the above question is a representation of a linear map $h : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the standard basis $B = \langle(1,0,0),(0,1,0),(0,0,1)\rangle$. Now let $D = \langle(1,1,1),(1,1,0),(1,0,0)\rangle$. Find

    (a) The representing matrix H for h with respective to D.

    (b) The representing matrix of the identity map with respective to the standard basis and D. That is $\text{Rep}_{B,D}(id)$.

    (c) The representing matrix of the identity map with respective to D and B. That is $\text{Rep}_{D,B}(id)$.

11. (4 pts) Use any method you have learned in this course to prove that a square matrix A is non-singular if and only if $|A| \neq 0$.

12. (3 pts each) Prove or disprove for each of the following (justify your answers):

    (a) Every triangular matrix can be diagonalized.

    (b) Every triangular matrix with distinct entries in the main diagonal can be diagonalized.

    (c) A square matrix without distinct eigenvalues can not be diagonalized.

    (d) The identity matrix of any size is similar only to itself.

13. (4 pts each) Find the eigenvalues and eigenspaces for each of the following matrices.

$$(a)\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (b)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

14. (4 pts) Prove or disprove: If A and B are both n by n matrices with $|A| = |B|$ and $rank(A) = rank(B)$, then A and B are similar.

15. (7 pts) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be linear map by rotation of angel θ counter-clock-wisely.

    (a) Find the representing matrix H with respect to the standard basis.

    (b) Find the eigenvalues and eigenspaces of H.