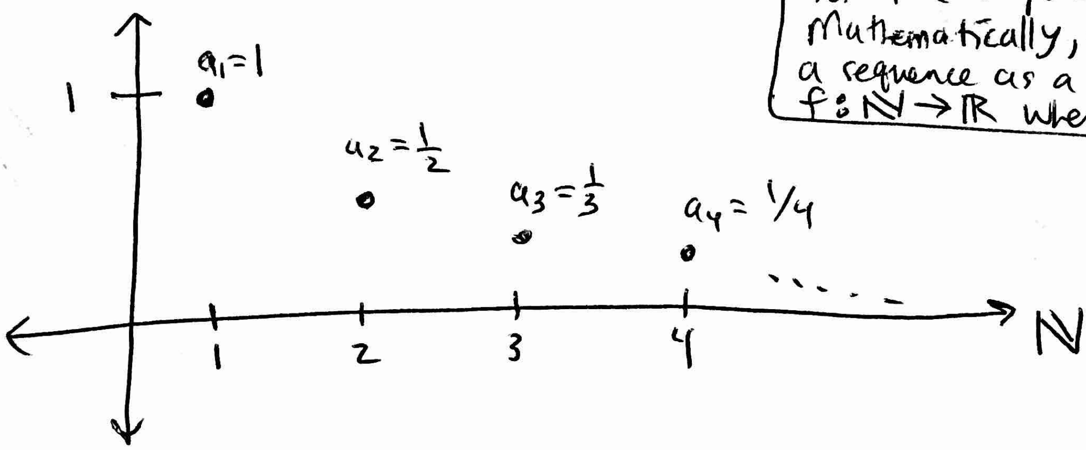
Limits of sequences

Def: A sequence of real numbers is a list of real numbers that is ordered using the natural numbers. ~~Mathematically, one can define the sequence as a function $f: \mathbb{N} \to \mathbb{R}$.~~
We write the nth term of the sequence as a_n.

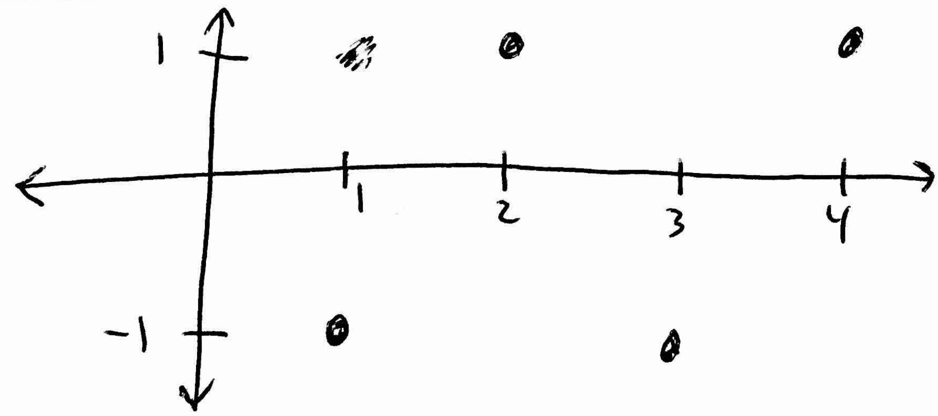
We sometimes write (a_n) or $(a_n)_{n=1}^{\infty}$ for the sequence. Mathematically, one can define a sequence as a function $f: \mathbb{N} \to \mathbb{R}$ where $f(n) = a_n$

Ex: $a_n = \frac{1}{n}$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$



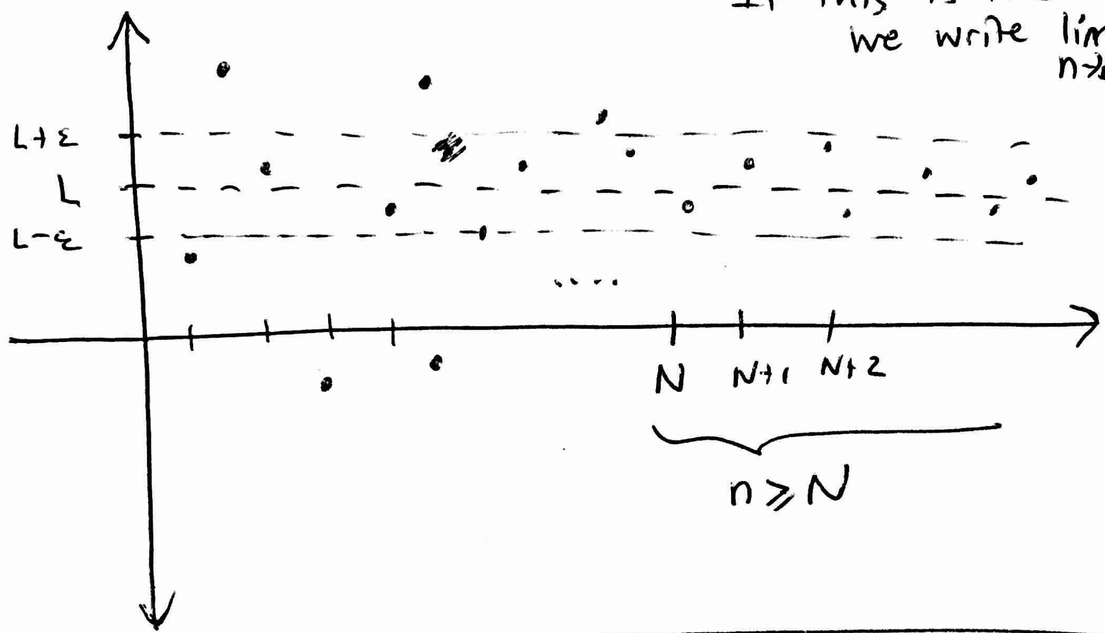
Ex: $a_n = (-1)^n$



Def: A sequence (a_n) in $\mathbb{R}$ is said to **converge** to a limit $L \in \mathbb{R}$ if

for every $\varepsilon > 0$, there exists a natural number N such that $|X_n - L| < \varepsilon$ if $n \geq N$.

If this is the case we write $\lim_{n\to\infty} a_n = L$.



Ex: $a_n = \frac{1}{n}$ ← We think that the limit $L = 0$.

~~If $n > 1000$, then~~

~~$|a_n - 0| = |\frac{1}{n}| = \frac{1}{n} < \frac{1}{1000} < \frac{\varepsilon}{1000} < \varepsilon$~~

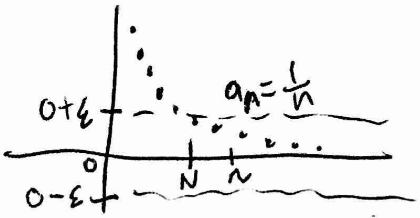
~~So a_n is within ε of L.~~

Let $\varepsilon > 0$

To make $|\frac{1}{n} - 0| < \varepsilon$

we need $\frac{1}{n} < \varepsilon$ iff $1/\varepsilon < n$.

Do some example ε and draw them



$\boxed{\text{Claim: } \lim_{n\to\infty} \frac{1}{n} = 0.}$

called this "useful fact" earlier

pf: Let $\varepsilon > 0$.

Let N be such that $\frac{1}{N} < \varepsilon$. If $n \geq N$,

then

$|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$

So, $|\frac{1}{n} - 0| < \varepsilon$ if $n \geq N$. ∎

We say a sequence (a_n) <u>diverges</u> if it does not converge.

<u>Ex:</u> Let $a_n = (-1)^n$.
Then (a_n) diverges.

<u>proof:</u> Suppose that (a_n) converges.
Then there exists $L \in \mathbb{R}$ with $\lim\limits_{n\to\infty} a_n = L$.

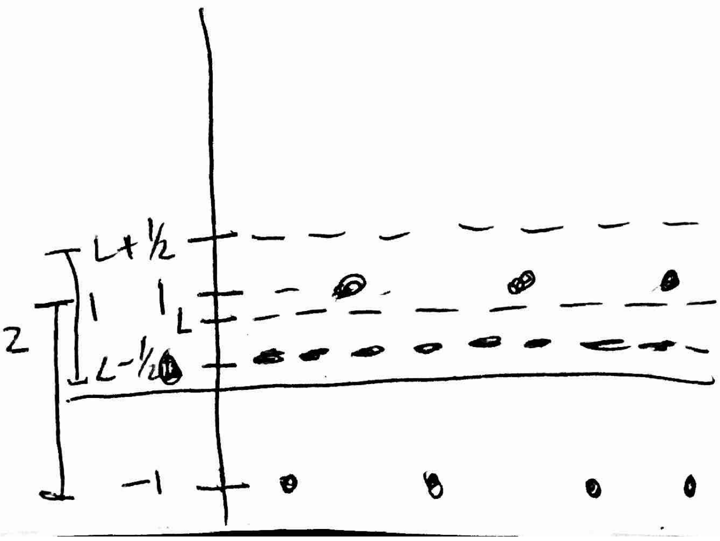
Let $\varepsilon = \frac{1}{2}$. Then there exists N where
$|a_n - L| < \frac{1}{2}$ for all $n \geq N$.

Let n_e be an even integer with $n_e \geq N$
and n_o be an odd integer with $n_o \geq N$.

Then, $|1 - L| = |a_{n_e} - L| < \frac{1}{2}$ ← ε
and $|(-1) - L| = |a_{n_o} - L| < \frac{1}{2}$

So, $$2 = |1-(-1)| = |1 - L + L - (-1)| \leq |1-L| + |L-(-1)| < \frac{1}{2} + \frac{1}{2} = 1.$$

So, $2 < 1$.
Contradiction. ∎



$$\boxed{\text{Ex: } \lim_{n\to\infty} \frac{n}{n+1} = 1}$$

Let $\varepsilon > 0$.

Then $\left|\frac{n}{n+1} - 1\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1}$.

There $\underbrace{\text{exists } N \in \mathbb{N} \text{ with } 0 < \frac{1}{N} < \varepsilon}_{\text{"Useful fact"}}$.

Then if $n \geq N$ we have

$$\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon. \quad \blacksquare$$

Thm: Limits are unique. That is, suppose (a_n) is a sequence that converges. Suppose L and L' are both limits of (a_n). Then $L = L'$.

proof: Let $\varepsilon > 0$.

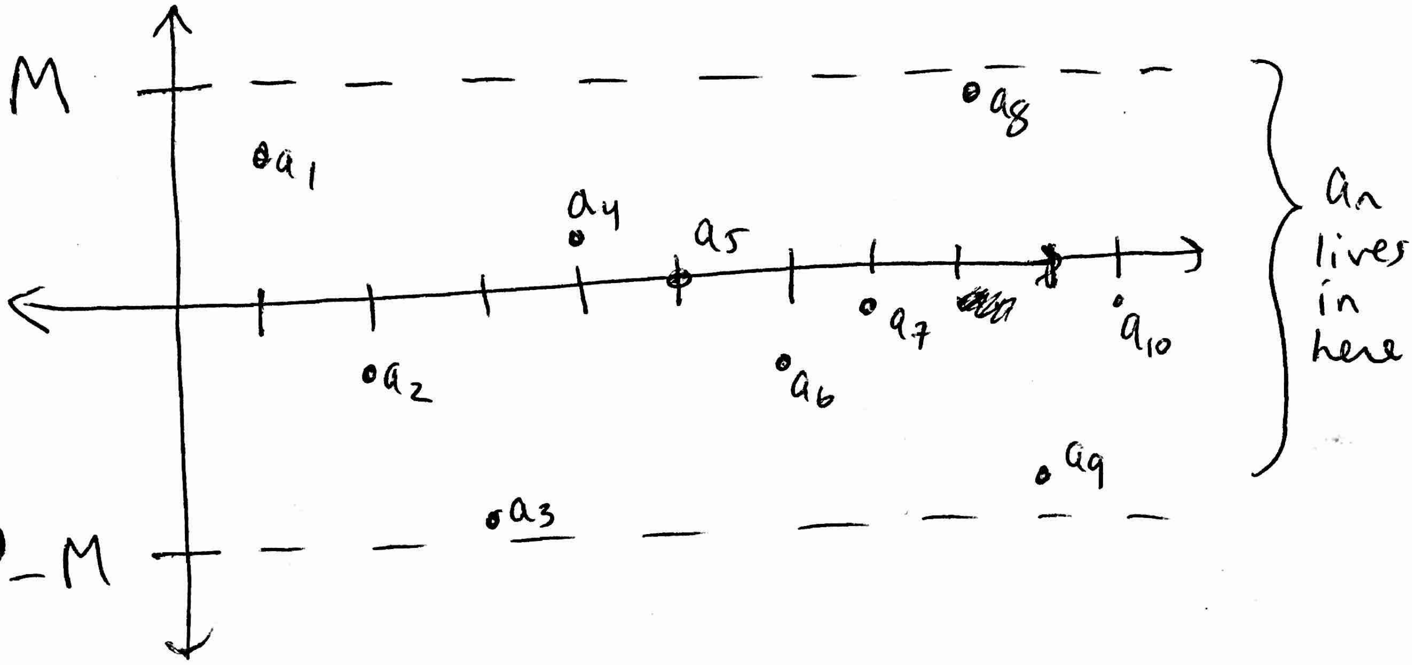
There exists N such that $|a_n - L| < \varepsilon/2$ for all $n \geq N$ and ~~there~~ exists M such that $|a_n - L'| < \varepsilon/2$ for all $n \geq M$.

Let $n = \max\{N, M\}$. So $n \geq N$ and $n \geq M$.

Then, $|L - L'| = |L - a_n + a_n - L'| \leq |L - a_n| + |a_n - L'| < \varepsilon$.

So, $|L - L'| \geq 0$ and is less than every possible positive real number. Thus, $|L - L'| = 0$. So, $L = L'$. $\blacksquare$

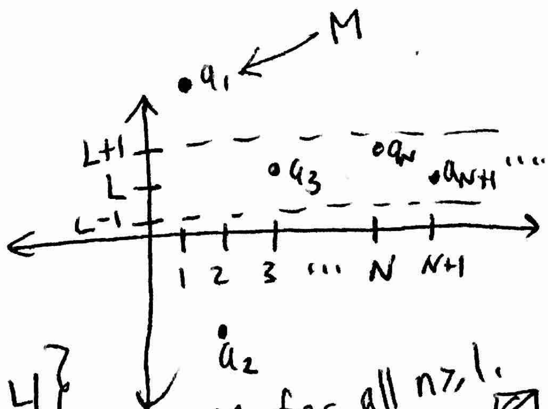
Def: A sequence (a_n) of real numbers is <u>bounded</u> if there exists a real number $M > 0$ such that $|a_n| \le M$ for all $n \ge 1$.



Thm: If (a_n) is a convergent sequence, Then (a_n) is bounded.

pf: Let $L = \lim_{n\to\infty} a_n$.

Let $\varepsilon = 1$. Then there exists N such that $|a_n - L| < 1$ for all $n \ge N$.



Thus if $n \ge N$ we have
$$|a_n| = |a_n - L + L| \le |a_n - L| + |L| \le 1 + |L|.$$

Let $M = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, 1+|L|\}$. Then $|a_n| \le M$ for all $n \ge 1$. So (a_n) is bounded. ∎

<u>Contrapositive:</u> If (a_n) is an unbounded sequence, then (a_n) diverges.

(a_n) is Unbounded means

For every $M > 0$ there exists some n with $|a_n| > M$.

<u>Ex:</u> (n^2) diverges.

<u>pf:</u> Let $M > 0$ be any real number.

Pick an integer n with $n > \sqrt{M}$.

Then, $n^2 > M$.

So, $|n^2| > M$.

Thus, (n^2) is unbounded.

So, (n^2) diverges. ∎

Thrm: Let (a_n) and (b_n) be convergent sequences. with limits A and B respectively. ① Then (a_n+b_n) converges to $A+B$. This says,

$$\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = \lim_{n\to\infty} (a_n+b_n)$$

② Also, (a_nb_n) converges to AB. This says,

$$\left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = \lim_{n\to\infty}(a_nb_n)$$

③ If $a_n \neq 0$ for all n ~~[illegible]~~ and $A \neq 0$, then $\left(\frac{1}{a_n}\right)$ converges to $\frac{1}{A}$. That is, $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{A}$.

proof:

① Let $\varepsilon > 0$. Since a_n converges to A, there exists $N_1 > 0$ such that $|a_n - A| < \varepsilon/2$ for all $n \geq N_1$. Since b_n converges to B, there exists $N_2 > 0$ such that $|b_n - B| < \varepsilon/2$ for all $n \geq N_2$. Let $M = \max\{N_1, N_2\}$. Then if $n \geq M$ we have $|(a_n+b_n) - (A+B)| = |a_n - A + b_n - B|$

$$\leq |a_n - A| + |b_n - B| < \varepsilon/2 + \frac{\varepsilon}{2} = \varepsilon.$$

② Let $\varepsilon > 0$. Consider

$$|a_n b_n - AB| = |a_n b_n - b_n A + b_n A - AB|$$
$$\leq |a_n b_n - b_n A| + |b_n A - AB|$$
$$= |b_n||a_n - A| + |A||b_n - B|.$$

Since (b_n) is a convergent sequence, it is bounded. Hence there exists $K > 0$ such that $|b_n| < K$ for all $n \geq 1$. Choose N_1 such that $|a_n - A| < \frac{\varepsilon}{2K}$ for all $n \geq N_1$.

Further, choose N_2 such that $|b_n - B| < \underbrace{\frac{\varepsilon}{2(|A|+1)}}$ if $n \geq N_2$.

Note: We put $|A|+1$ in case $|A| = 0$

Let $M = \max\{N_1, N_2\}$.

Then if $n \geq M$ we have that

$$|a_n b_n - AB| \leq |b_n||a_n - A| + |A||b_n - B|$$
$$< K \cdot \frac{\varepsilon}{2K} + |A| \cdot \frac{\varepsilon}{2(|A|+1)}$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

3 HW. $\blacksquare$

$\boxed{\text{Mon~~O~~tone convergence Thm}}$

Def: Let (a_n) be a sequence of real numbers. We say that (a_n) is monotonically increasing if $a_n \leq a_{n+1}$ for all $n \geq 1$.

We say that (a_n) is monotonically decreasing if $a_{n+1} \leq a_n$ for all $n \geq 1$. We say that a sequence is monotone if it is either increasing or decreasing.

Ex: (n) is increasing.
$\left(\frac{1}{n}\right)$ is decreasing.

Montone convergence Thm: A monotone sequence of real numbers is convergent if and only if it is bounded.

Furthermore:

① If (a_n) is a bounded increasing sequence then $\lim_{n\to\infty} a_n = \sup\{a_n \mid n = 1, 2, 3, \ldots\}$

② If (a_n) is a bounded decreasing sequence then $\lim_{n\to\infty} a_n = \inf\{a_n \mid n = 1, 2, 3, \ldots\}$

proof: We do 1.

Let (a_n) be a bounded increasing sequence.

~~We know from a previous theorem that if (a_n) converges, then it is bounded. So, suppose now that (a_n) is bounded.~~

Then there exists $M > 0$ such that $|a_n| < M$ for all $n \geq 1$.

Since (a_n) is bounded, the set $S = \{a_n \mid n = 1, 2, 3, \ldots\}$
$= \{a_1, a_2, a_3, a_4, \ldots\}$

is bounded from above. By the completeness property of $\mathbb{R}$ we know that $L = \sup(S)$ exists. We now show that L is the limit of (a_n).

Let $\varepsilon > 0$. Since L is the supremum of S, $L - \varepsilon$ is not an upper bound for S.

Thus, there exists some N such that $a_N > L - \varepsilon$. Since (a_n) is an increasing sequence we know that $a_N \leq a_n$ for all $n \geq N$. Hence if $n \geq N$, then

$$L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon.$$

So, $|a_n - L| < \varepsilon$ for all $n \geq N$. Thus, (a_n) converges to L. ∎

Application: Finding square roots.

Note: Formula comes from Newton's method.

Let $a > 0$. We will construct a sequence (a_n) of real numbers that converges to $\sqrt{a}$.

Let $a_1 > 0$ be arbitrary.

Define $a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right)$ for $n = 1, 2, 3, 4, 5, \ldots$

Note: $a_n > 0$ for all $n \geq 1$

Claim 1: $a_n^2 \geq a$ for $n \geq 2$

Note that $a_n^2 - 2a_{n+1}a_n + a = 0$ for $n \geq 1$.

Hence, $x^2 - 2a_{n+1}x + a = 0$ has a real root.

So, the discriminant must be non-negative.

$$\boxed{x = \frac{2a_{n+1} \pm \sqrt{4a_{n+1}^2 - 4a}}{2}}$$

So, $4a_{n+1}^2 - 4a \geq 0$.

So, $a_{n+1}^2 \geq a$ for $n \geq 1$. ▨ (claim 1)

Claim 2: (a_n) is ultimately decreasing

When $n \geq 2$ we have that

$$a_n - a_{n+1} = a_n - \frac{1}{2}\left(a_n + \frac{a}{a_n}\right) = \frac{a_n^2 - a}{2a_n} \geq 0.$$

(by claim 1 since $n \geq 2$)

Hence, $a_n \geq a_{n+1}$ for all $n \geq 2$. ▨ (claim 2)

The Monotone convergence theorem tells us that (a_n) has a limit. Let $A = \lim_{n\to\infty} a_n$. Since $a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right)$ we have that $A = \frac{1}{2}\left(A + \frac{a}{A}\right)$.

Solving this gives $A^2 = a$. So, $A = \sqrt{a}$.

Find $\sqrt{a}$, $a>0$
$a_1 =$ arbitrary >0
$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right)$$
Newton's method

Error bound in above calculation:

Note $a_n \geq \sqrt{a}$ for all $n \geq 2$.

So, $\underbrace{a_n \geq \sqrt{a}}_{\text{because } a_n \geq \frac{a}{\sqrt{a}} = \sqrt{a}} \geq \frac{a}{a_n}$

Thus,
$$0 \leq a_n - \sqrt{a} \leq a_n - \frac{a}{a_n} = \frac{(a_n^2 - a)}{a_n} \quad \text{for } n \geq 2$$

Thus $\boxed{|a_n - \sqrt{a}| \leq \frac{a_n^2 - a}{a_n},\ n \geq 2}$

Ex: Let $a=2$. Let's estimate $\sqrt{2}$.

Let $a_1 = 1$.

$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$	Error bound $\frac{a_n^2 - a}{a_n}$
$a_2 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} = 1.5$	$\frac{a_2^2 - 2}{a_2} = \frac{\left(\frac{3}{2}\right)^2 - 2}{\frac{3}{2}} = \frac{1}{6} \approx 0.1666...$
$a_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{\left(\frac{3}{2}\right)}\right) = \frac{17}{12} \approx 1.416666...$	$\frac{a_3^2 - 2}{a_3} = \frac{1}{204} \approx 0.0049019607...$
$a_4 = \frac{577}{408} \approx 1.414215686...$	$\frac{a_4^2 - 2}{a_4} = \frac{1}{235416} \approx 0.0000042477996...$
$a_5 = \frac{665857}{470832} \approx 1.41421356237469...$	$\frac{1}{313{,}506{,}783{,}024} \approx 3.1897 \times 10^{-12}$

Application: Bolzano-Weierstrass Theorem

Def: Let (x_n) be a sequence of real numbers. Let $n_1 < n_2 < n_3 < \dots$ be a strictly increasing sequence of natural numbers. Then the sequence $x_{n_1}, x_{n_2}, x_{n_3}, \dots$ is called a subsequence of (x_n)

Monotone subsequence Theorem:

If (x_n) is a sequence of real numbers, then there is a subsequence of (x_n) that is monotone.

EX: $(1/n)$
subsequence
$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

proof: We say that the mth term x_m is a "peak" of our sequence if $x_m \geq x_n$ for all $n \geq m$.

case 1: (x_n) has infinitely many peaks

In this case, we list the peaks by increasing subscripts $x_{m_1}, x_{m_2}, x_{m_3}, \dots$ (ie, $m_1 \leq m_2 \leq m_3 \leq \dots$)

Since each term is a peak we have

$$x_{m_1} \geq x_{m_2} \geq x_{m_3} \geq \dots$$

Therefore, the sequence $x_{m_1}, x_{m_2}, x_{m_3}, \dots$ is a monotone decreasing subsequence of (x_n) and hence montonic,

case 2: (x_n) has finitely many (possibly zero) peaks
~~let $s_1=1$ if there are no peaks. Otherwise~~ define s_1 as follows
Let these peaks be listed by increasing subscripts: $x_{m_1}, x_{m_2}, \dots, x_{m_r}$.

Let $s_1 = m_r + 1$ so that x_{s_1} is the term immediately after the last peak of our sequence.
So, x_{s_1} is not a peak. Thus, there exists $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$.

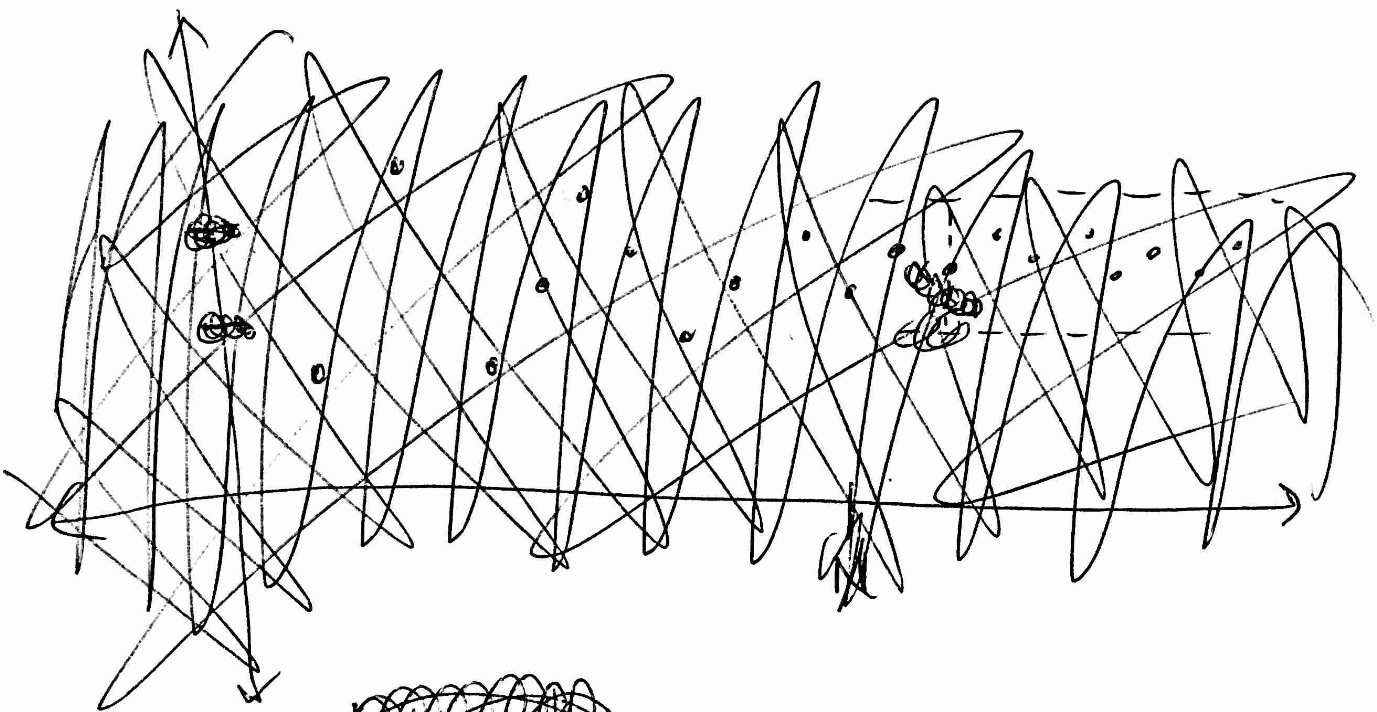
Since x_{s_2} is not a peak there exists $s_3 > s_2$ with $x_{s_2} < x_{s_3}$.
Continuing in this way we get an increasing subsequence $x_{s_1} < x_{s_2} < x_{s_3}, \dots$ which is montone. ▨

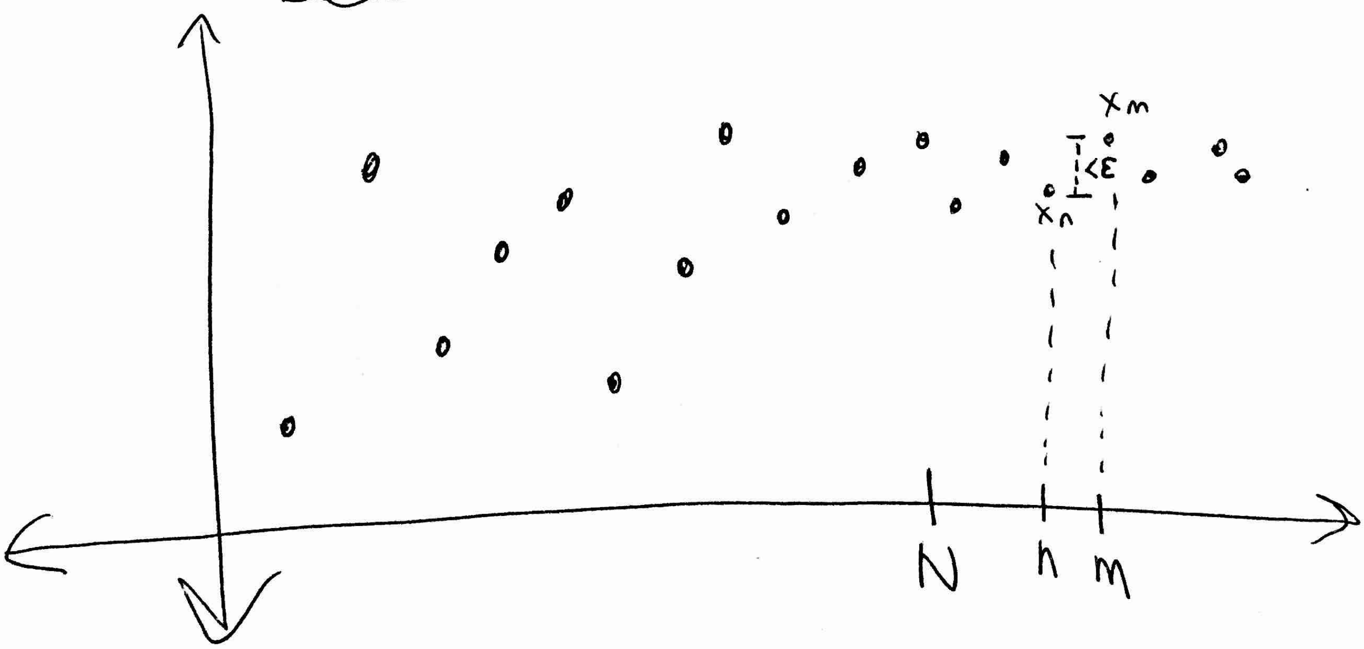
Bolzano Weierstrass: Let (x_n) be a bounded sequence of real numbers. Then there exists a subsequence that converges.

proof: By the Montone subsequence Theorem, there exists a subsequence (x_{n_k}) that is montonic. Since (x_{n_k}) is a subsequence of a bounded sequence we must have that (x_{n_k}) is also bounded. By the montonore convergence theorem, (x_{n_k}) converges. ▨

Ex: $((-1)^n)$ is a bounded sequence. Two convergent subsequences:
$1, 1, 1, 1, 1, 1, \dots$
$-1, -1, -1, -1, -1, \dots$

__Def:__ Let (a_n) be a sequence of real numbers. We say that (a_n) is a __Cauchy sequence__ if for every $\varepsilon > 0$, there exists $N > 0$ such that if $n, m \geq N$, then $|a_n - a_m| < \varepsilon$.





Ex: $\left(\frac{1}{n}\right)$ is Cauchy.

Let $\varepsilon > 0$.
Pick N so that $N > \frac{2}{\varepsilon}$.
Note that $\frac{1}{N} < \frac{\varepsilon}{2}$.
Then if $n, m \geq N$, then

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

~~Lemma: Let (a_n) be a Cauchy sequence of real numbers. Then (a_n) is bounded.~~

~~Proof: HW.~~

Theorem: Let (a_n) be a sequence of real numbers. Then (a_n) converges iff (a_n) is Cauchy.

Proof: ($\Rightarrow$) Suppose (a_n) converges to $L \in \mathbb{R}$.
Let $\varepsilon > 0$.
Pick $N > 0$ so that if $n \geq N$, then $|a_n - L| < \frac{\varepsilon}{2}$.
Then if $n, m \geq N$, then

$$\begin{aligned}|a_n - a_m| = |a_n - L + L - a_m| &\leq |a_n - L| + |L - a_m| \\ &= |a_n - L| + |a_m - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

($\Leftarrow$) Suppose that (a_n) is a Cauchy sequence.

HW: If (a_n) is Cauchy, then (a_n) is bounded.

So, (a_n) is bounded.
By Bolzano-Weierstrass, (a_n) has a convergent subsequence (a_{n_k}) that converges to $L \in \mathbb{R}$.

Let $\varepsilon > 0$.
Since (a_n) is Cauchy, there exists $N > 0$ such that if $k, \ell \geq N$, then $|a_k - a_\ell| < \frac{\varepsilon}{2}$.
Since (a_{n_k}) converges to L, there exists some $M > N$ so that $|a_M - L| < \frac{\varepsilon}{2}$.

Suppose that $n \geq N$.
Then,

$$
\begin{aligned}
|a_n - L| &= |a_n - a_M + a_M - L| \\
&\leq |a_n - a_M| + |a_M - L| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare
\end{aligned}
$$

Note: ($\Leftarrow$) depended on the completeness property of $\mathbb{R}$. It's not iff in noncomplete fields.