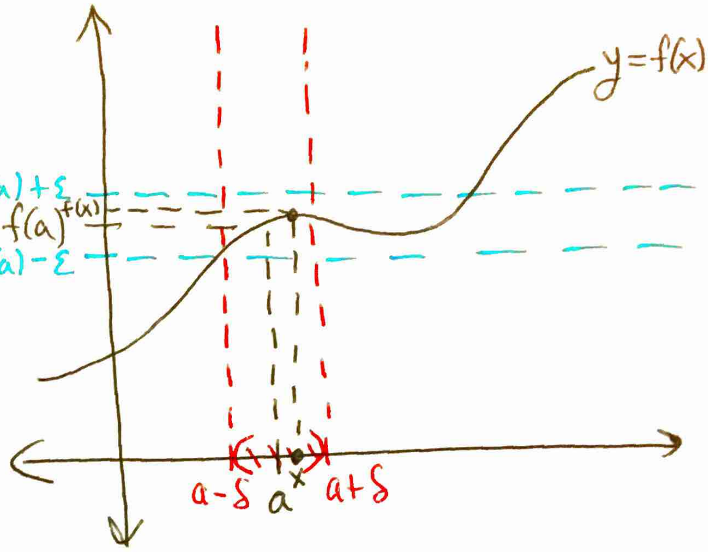
Continuity

Def: Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, and $a \in A$. We say that f is continuous at a if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x-a| < \delta$ and $x \in A$, then

$$|f(x) - f(a)| < \varepsilon.$$



If $B \subseteq A$ and f is continuous at all $b \in B$ then we say that f is continuous on B.

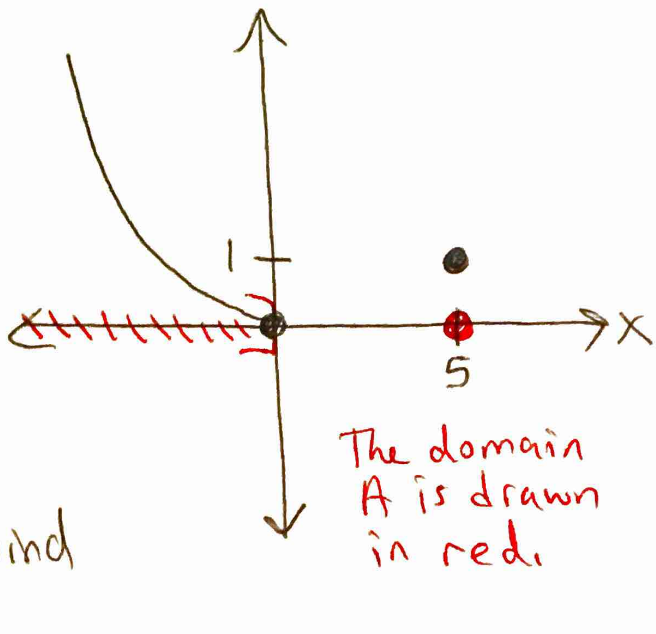
Consider the function

$$g(x) = \begin{cases} x^2, & x \le 0 \\ 5, & x = 1 \end{cases}$$

that has domain

$$A = (-\infty, 0] \cup \{5\}$$

Keep this function in mind for the following note.



The domain A is drawn in red.

Note: Let $f: A \to \mathbb{R}$ with $a \in \mathbb{R}$.

case 1: Suppose that a is a limit point of A

Then we may consider $\lim_{x \to a} f(x)$.

Looking at the definition of continuity,
f is continuous at a iff

1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

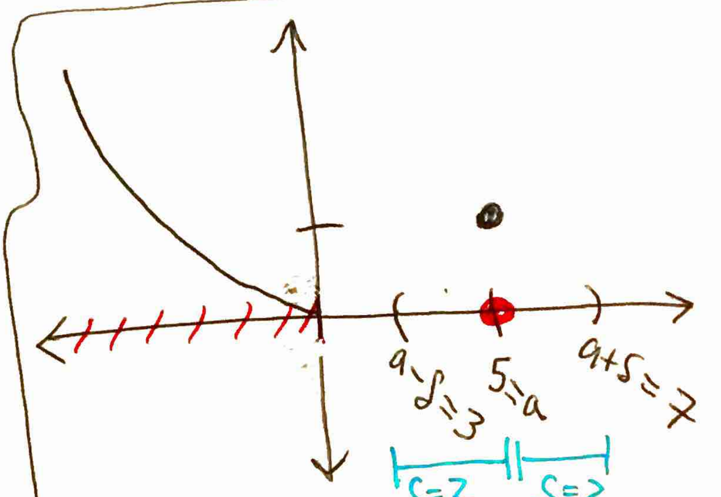
This case is $a \in (-\infty, 0]$ in the previous $g(x)$ example

case 2: Suppose that a is not a limit point of A.

This case is the $g(x)$ example with $a=5$

Then there exists $\delta > 0$ so that
$(a-\delta, a+\delta) \cap A = \{a\}$.

Then if $x \in A$ and
$|x-a| < \delta$ we have
that $x = a$.



So,
$$|f(x)-f(a)| = |f(a)-f(a)| = 0 < \varepsilon \text{ for any } \varepsilon > 0.$$

So in this case f is continuous at a.

~~Continuity~~

~~**Def:** Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, and $a \in A$. We say that f is continuous at a if $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = f(a)$.~~

~~Let $B \subseteq A$. We say that f is continuous on B if f is cont. at every $a \in B$.~~

Ex: Let $f(x) = x^2$. Since every $a \in \mathbb{R}$ is a limit pt of f, we just need to show that $\lim_{x \to a} f(x) = f(a)$ to show that f is cts at a.

We saw in an earlier lecture that $\lim_{x \to 2} x^2 = 4$.

So, $f(x) = x^2$ is continuous at $a = 2$.

Let's show $f(x) = x^2$ is continuous on all of $\mathbb{R}$.

Let $a \in \mathbb{R}$.

Let $\varepsilon > 0$.

Note that

$$|f(x) - f(a)| = |x^2 - a| = |x+a||x-a|.$$

If $0 < |x-a| < 1$, then

$$|x+a| = |x \underbrace{- a + a}_{0} + a| \le |x-a| + 2|a| < 1 + 2|a|.$$

Let $\delta = \min\left\{1, \frac{\varepsilon}{1+2|a|}\right\}$. If $0 < |x-a| < \delta$, then

$$|f(x) - f(a)| = |x+a||x-a| < (1+2|a|)\frac{\varepsilon}{1+2|a|} = \varepsilon. \quad \blacksquare$$

Thm: Let $A \subseteq \mathbb{R}$. Let $a \in A$ and $\alpha \in \mathbb{R}$.
Suppose that f and g are continuous at a.
Then αf, $f+g$, $f-g$, $f \cdot g$ are all continuous at a.
If $f(a) \neq 0$, then $\frac{1}{f}$ is continuous at a.

pf: If a is not a cluster point then all the functions are continuous at a.
Suppose a is a cluster point of A, then the thm follows by the theorems on limits. For example, since f and g are continuous at a we have $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$.

So, $\lim_{x \to a} (f+g)(x) = \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

$= f(a) + g(a)$

$= (f+g)(a)$.

$\boxtimes$

40'

Thm: Let $A, B \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$. If f is continuous at some point $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f : A \to \mathbb{R}$ is continuous at c.

proof: Let $\varepsilon > 0$.

Since g is continuous at $f(c)$, there exists $\delta_1 > 0$ where if $y \in B$ and $|y - f(c)| < \delta_1$ then $|g(y) - g(f(c))| < \varepsilon$.

Since f is continuous at c, there exists $\delta_2 > 0$ such that if $x \in A$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \delta_1$.

Since $f(A) \subseteq B$, we have that if $x \in A$ and $|x - c| < \delta$, then $f(x) \in B$ and $|f(x) - f(c)| < \delta_1$. So, if $x \in A$ and $|x - c| < \delta$, then

$$|g(f(x)) - g(f(c))| < \varepsilon.$$

∎

The Intermediate Value Theorem

Let f be continuous on $[a,b]$.
Suppose that $f(a) < f(b)$.
For each $d \in \mathbb{R}$ with $f(a) < d < f(b)$
there exists $c \in (a,b)$ with $f(c) = d$

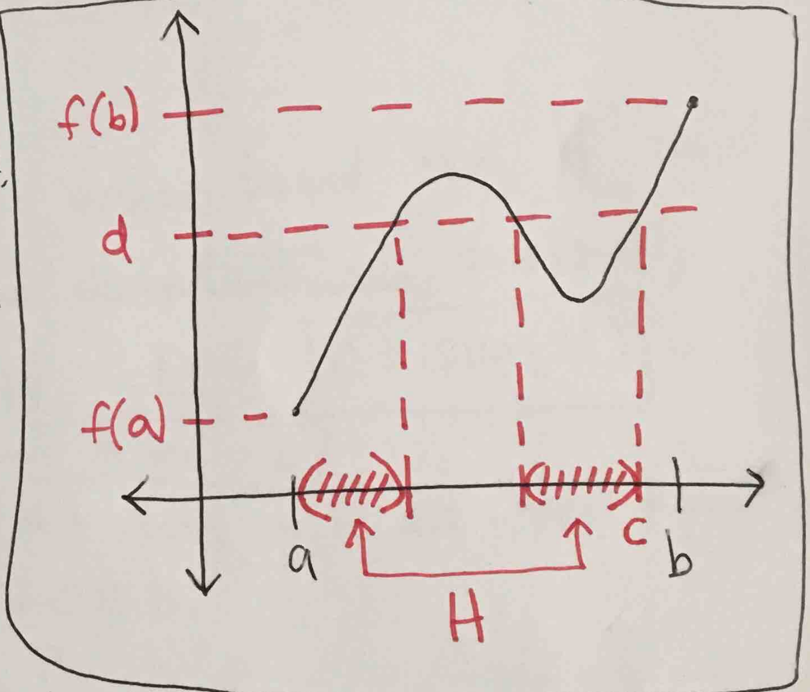
proof:

Let $H = \{x \in (a,b) \mid f(x) < d\}$

① We first show that $\sup(H)$ exists

Let's show that $H \neq \emptyset$ first.

Let $\varepsilon = d - f(a) > 0$.
Since f is continuous at a, there exists $\delta > 0$ where if $x \in [a,b]$ and $|x-a| < \delta$ then



$|f(x) - f(a)| < \varepsilon = d - f(a)$.

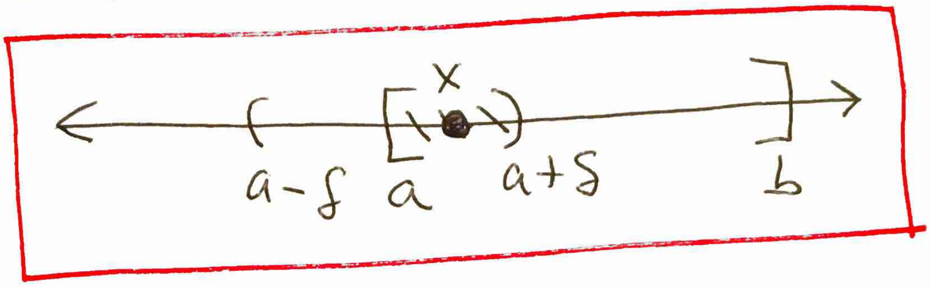
We may assume that $\delta < b-a$ by shrinking δ if necessary

So we may assume that $a + \delta < b$.

Note that $x \in [a,b]$ and $|x-a| < \delta$
is equivalent to $a \leq x < a+\delta$.

So, if

$a \leq x < a+\delta$



then

$f(x) - f(a) \leq |f(x) - f(a)| < d - f(a)$

That is, if $a \leq x < a+\delta$, then $f(x) < d$.

So, $(a, a+\delta) \subseteq H$.

So, $H \neq \emptyset$.

Since b is an upper bound for H and
$H \neq \emptyset$, by the completeness axiom,
$\sup(H)$ exists. Let $\boxed{c = \sup(H)}$.

$\boxed{②\text{ We now show that } a < c < b}$

Since $c = \sup(H)$ and b is an upper bound of H,
we know that $c \leq b$.

Why is $c \neq b$?

Since f is continuous at b, there exists
$\delta' > 0$ where if $x \in [a,b]$ and $|x-b| < \delta'$
then $|f(x) - f(b)| < f(b) - d$.

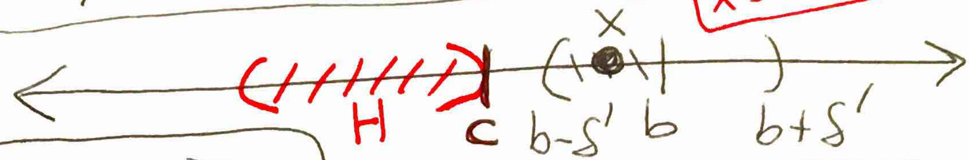
So, if $x \in [a,b]$ and $|x-b| < \delta'$ then

$-(f(b)-d) < f(x) - f(b) < f(b) - d$.

That is, if $x \in (b-\delta', b]$

then $-(f(b)-d) < f(x) - f(b)$.

none of these x's are in H



So c is over here

So, if $x \in (b-\delta', b]$

then $d < f(x)$.

So, if $x \in (b-\delta', b]$, then $x \notin H$.

Thus, $c \leq b - \delta' < b$.

That is $\boxed{c < b}$.

Why is $a < c$?

Since $H \neq \emptyset$, there exists $x_0 \in H$.

So, $a < x_0$.

Then $a < x_0 \leq c$ since $c = \sup(H)$.

Thus, $\boxed{a < c}$

Therefore $\boxed{a < c < b}$.

③ We now show that $f(c) = d$

We do this by showing that $f(c) < d$ and $d < f(c)$ cannot occur.

<u>case (i)</u> Suppose $f(c) < d$.

We show that this leads to a contradiction.

Since $f(c)<d$, if we set $\varepsilon = d - f(c)$ we get $\varepsilon > 0$. 42''

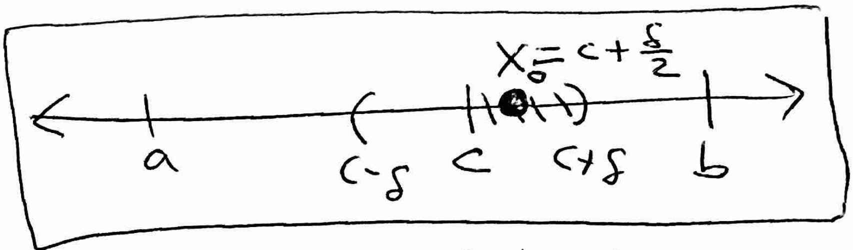
Since $a<c<b$, f is continuous at c.
Thus, there exists $\delta>0$ where if $x\in[a,b]$ and $|x-c|<\delta$ then

$$f(x)-f(c) \le |f(x)-f(c)| < \varepsilon = d-f(c)$$

<u>Note:</u> We may assume that $\delta < b-c$ by shrinking δ if necessary

Thus, in particular, if $c \le x < c+\delta$ then $f(x)-f(c) < d-f(c)$,
So if $c\le x<c+\delta$ then $f(x)<d$.

So for example if $x_0 = c+\frac{\delta}{2}$ then $c<x_0$ and $f(x_0)<d$.

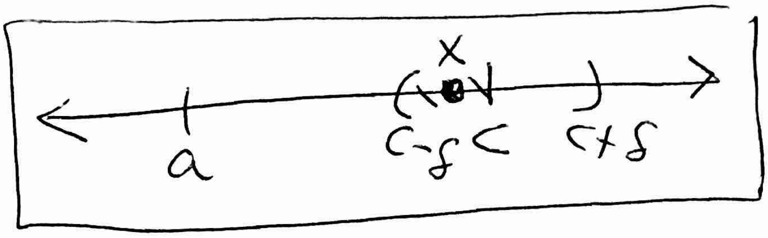


This contradicts that $c=\sup(H)$ since $x_0\in H$ and $c<x_0$.

<u>case (ii)</u> Suppose $f(c)>d$.
Then $f(c)-d>0$.
Set $\varepsilon = f(c)-d>0$.
Since f is continuous at c there exists $\delta>0$ where if $x\in[a,b]$ and $|x-c|<\delta$ then $|f(x)-f(c)|<\varepsilon$.
We may assume that $\delta<c-a$ by shrinking δ if necessary.

So, in particular, if $x \in (c-\delta, c]$ then $f(c)-f(x) \leq |f(x)-f(c)| < f(c)-d$.
So, if $a < c-\delta < x \leq c$, $d < f(x)$.

So,
$H \cap (c-\delta, c] = \phi$.



However, since $c = \sup(H)$, by the useful sup/inf fact, there exists $h \in H$ with $c-\delta < h \leq c$. Contradiction.

Thus, $f(c) = d$.