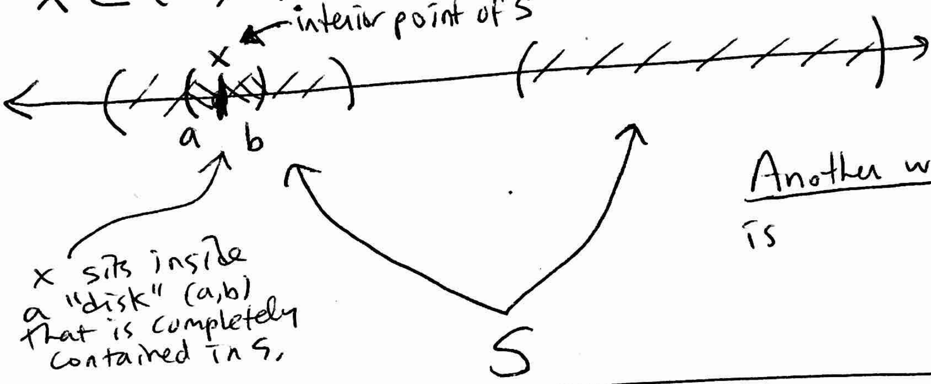
Open and Closed subsets of $\mathbb{R}$

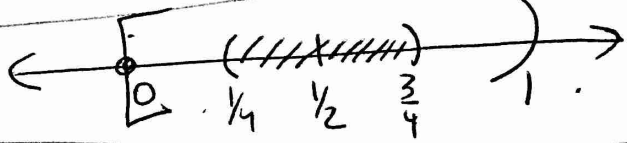
Def: Let $S \subseteq \mathbb{R}$.

① We say that $x \in \mathbb{R}$ is an <u>interior point</u> of S if there exists an interval (a,b) of $\mathbb{R}$ where $x \in (a,b)$ and $(a,b) \subseteq S$.



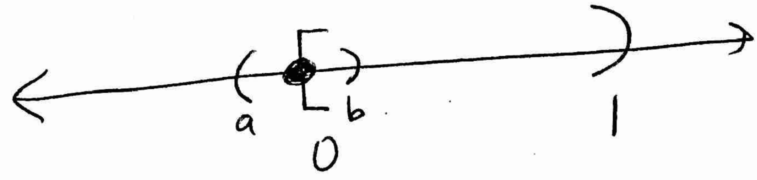
Another way: $S \subseteq \mathbb{R}$ is

Ex: Let $S = [0,1)$.



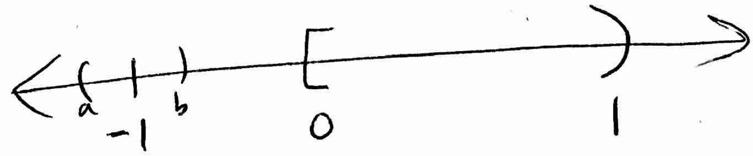
$1/2$ is an interior point

0 is not an interior point.



Any open interval around 0 is not a subset of $[0,1)$.

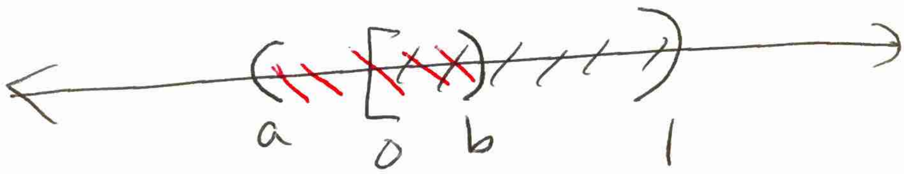
-1 is not an interior point



Same reason as with 0.

<u>Def:</u> We say that $S \subseteq \mathbb{R}$ is <u>open</u> if every x in S is an interior point of S.

<u>Ex:</u> $[0,1)$ is not open since $0 \in [0,1)$ but 0 is not an interior point of $[0,1)$ since every interval (a,b) around 0 is not contained in $[0,1)$.



<u>Ex:</u> Let $a,b \in \mathbb{R}$ with $a<b$. Then (a,b) is open.

<u>pf:</u> Let $x \in (a,b) = S$. Then choosing the interval (a,b), we have

$$x \in (a,b) \subseteq (a,b) = S$$

▨

Note: Another way to say open: $S \subseteq \mathbb{R}$ is open iff for every $z \in S$ there exists $\varepsilon > 0$ so that $(z-\varepsilon, z+\varepsilon) \subseteq S$.

Def: Let $S \subseteq \mathbb{R}$. We say that S is closed if ~~$S^c = \mathbb{R} \setminus S$ is open~~ ~~Here S is the complement of S~~ the complement of S

$$S^c = \mathbb{R} \setminus S = \{x \in \mathbb{R} \mid x \notin S\}$$

is open.

Ex: $S = (-\infty, 1] \cup [5, \infty)$

Then $S^c = (1,5)$ is open.

So, S is closed.

HW:
- $\mathbb{R}$ is open
- ϕ is open
- If $a \in \mathbb{R}$, then (a, ∞) is open.
- If $a \in \mathbb{R}$, then $(-\infty, a)$ is open.
- If A and B are open, then $A \cup B$ is open.

Ex: Let $a, b \in \mathbb{R}$ with $a < b$. Then $[a,b]$ is closed.

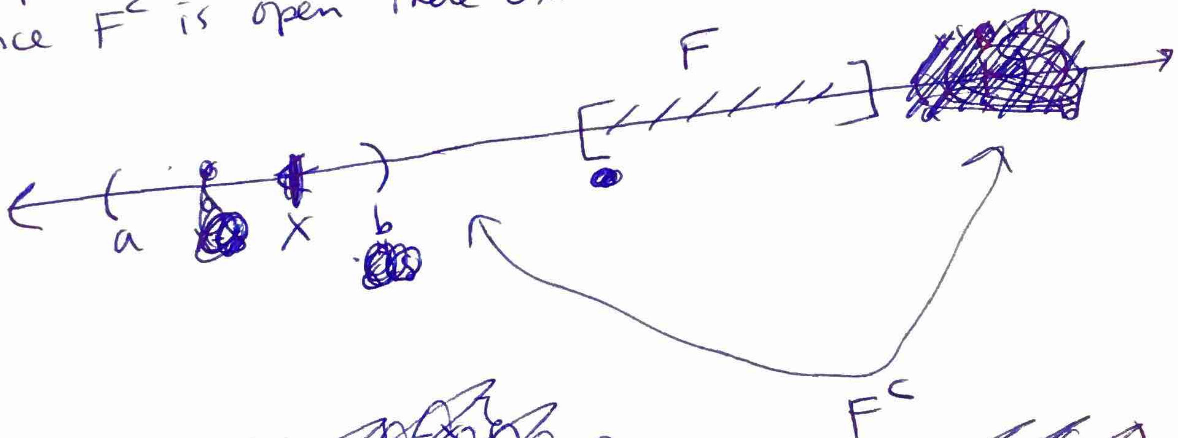
pf: $[a,b]^c = (-\infty, a) \cup (b, \infty)$ which is open by HW. ∎

Mention: $[0,1)$ not open and not closed. Example of such a set.

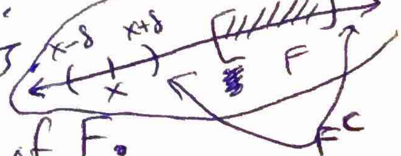
Thm: Let $F \subseteq \mathbb{R}$.
F is closed iff F contains all its limit points.

> **Recall:** $a \in \mathbb{R}$ is a limit point of F if for every $\delta > 0$, there exists $x \in F$ such that $x \neq a$ and $|x - a| < \delta$.

pf: ($\Longrightarrow$) Suppose F is closed.
Then F^c is open.
Let $x \in \mathbb{R}$ be a limit point of F.
Let's show that $x \notin F^c$.
Suppose $x \in F^c$.
Since F^c is open there exists $a < b$ with $x \in (a,b) \subseteq F^c$.



This would contradict the fact that x is a limit point of F since (a,b) is an interval around x that contains no points of F other than x.



($\Longleftarrow$) Suppose F contains all its limit points.
Let's show F^c is open so that F is closed.
Let $x \in F^c$. Since $x \notin F$ it is not a limit point of F.
So, there exists an interval $(x-\delta, x+\delta)$ so that $(x-\delta, x+\delta) \cap F = \emptyset$.
So, $x \in (x-\delta, x+\delta) \subseteq F^c$.
So, F^c is open. ■