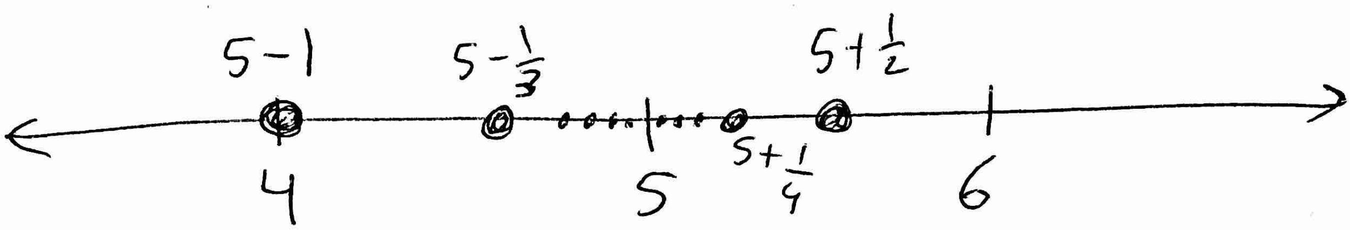
Test 1, Math 4650, Fall 2018 Name: Solutions

1. [10 points - 5 each] Find the supremum and infimum of each set if they exist.

First draw a picture of the set or list several elements of the set to get an idea of what's going on. Also, this will help me give you partial credit if you're end result isn't fully correct.

(a) $Y = \left\{ 5 + \frac{(-1)^n}{n} \;\middle|\; n = 1, 2, 3, 4, \ldots \right\}$

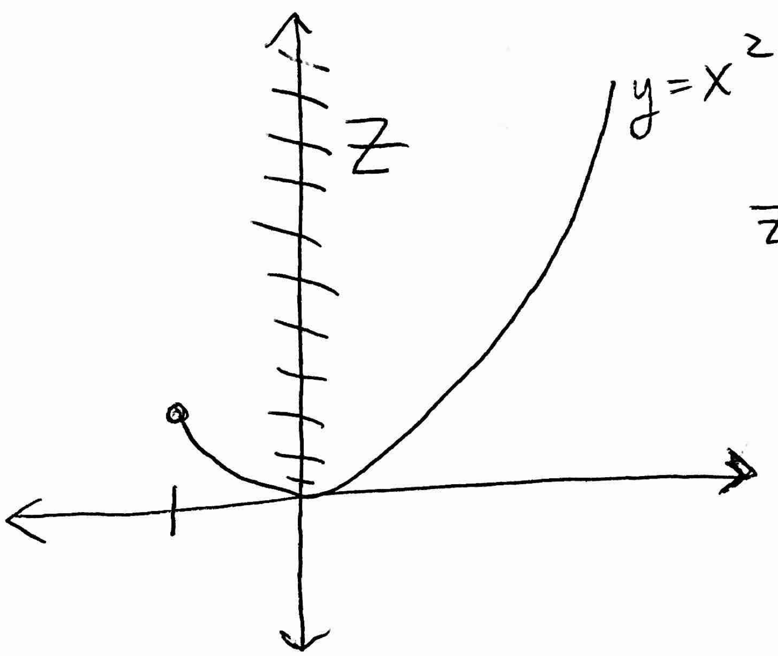
$Y = \left\{ 5 - \frac{1}{1}, 5 + \frac{1}{2}, 5 - \frac{1}{3}, 5 + \frac{1}{4}, 5 - \frac{1}{5}, \ldots \right\}$



$\inf(Y) = 4$

$\sup(Y) = 5 + \frac{1}{2} = 5.5$

(b) $Z = \{x^2 \in \mathbb{R} \mid -1 \leq x\}$



$Z = [0, \infty)$

$\inf(Z) = 0$

$\sup(Z)$ does not exist

2. [10 points] Prove that

$$\lim_{n\to\infty} \frac{-2n^5-1}{1+n^5} = -2$$

On this problem, you must prove the result using the definition of limit. No theorems.

Let $\varepsilon > 0$.

Note that

$$\left| \frac{-2n^5-1}{1+n^5} - (-2) \right| = \left| \frac{-2n^5-1+2+2n^5}{1+n^5} \right| =$$

$$= \left| \frac{1}{1+n^5} \right| \underset{\boxed{n \geq 1}}{=} \frac{1}{1+n^5}$$

And $\frac{1}{1+n^5} < \varepsilon$ iff $\frac{1}{\varepsilon} < 1+n^5$ iff $\frac{1}{\varepsilon} - 1 < n^5$

iff $\sqrt[5]{\frac{1}{\varepsilon}-1} < n$.

Set $N > \sqrt[5]{\frac{1}{\varepsilon}-1}$.

If $n \geq N$, then from above we have that

$$\left| \frac{-2n^5-1}{1+n^5} - (-2) \right| = \frac{1}{1+n^5} < \varepsilon.$$

$\blacksquare$

3. [10 points - 5 each] True or False. If True, give a proof. If False, give a specific example showing that the statement can be false.

(a) If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$. $\boxed{\text{True}}$

Let $\varepsilon > 0$.

Since $\lim_{n\to\infty} |a_n| = 0$, there exists $N > 0$ such that if $n \geq N$, then $\big|\, |a_n| - 0 \,\big| < \varepsilon$.

So, if $n \geq N$, then $\big|\, |a_n| \,\big| < \varepsilon$.

That is, if $n \geq N$, then $|a_n| < \varepsilon$.

So, if $n \geq N$, then $|a_n - 0| < \varepsilon$.

Thus, $\lim_{n\to\infty} a_n = 0$.

(b) If (a_n) is a bounded sequence of real numbers, then (a_n) converges.

False

Ex: $a_n = (-1)^n$

This sequence is bounded but does not converge.

4. [10 points] PICK ONE. If you do both, then I will grade A.

A. Let (a_n) be a convergent sequence of real numbers. Prove that (a_n) is bounded.
(*On this problem, you are only allowed to use the definition of limit to prove the result.*)

B. Suppose that (a_n) and (b_n) are sequences of real numbers such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ where A and B are real numbers. Prove that $\lim_{n\to\infty} a_n b_n = AB$.
(*On this problem, you can use the definition of limit, and results about boundedness of sequences to prove the result.*)

These are in the lecture notes.
We did them both in class.

5. [10 points] Let S a non-empty subset of the real numbers that is bounded from below. Let c be a real number. Define the set

$$S_c = \{x + c \mid x \in S\}$$

Prove that $\inf(S_c) = \inf(S) + c$.

$\boxed{\text{Method 1}}$

Let $a = \inf(S)$ which exists since $S \neq \emptyset$ is bounded from below.

① Then $a \leq x$ for all $x \in S$.
Thus, $a + c \leq x + c$ for all $x \in S$.
So, $a + c$ is a lower bound for S_c.

② Suppose that b is another lower bound for S_c.
Then $b \leq x + c$ for all $x \in S$.
Thus, $b - c \leq x$ for all $x \in S$.
Since a is the greatest lower bound for S we have that $b - c \leq a$.
So, $b \leq a + c$.
Hence $a + c$ is the greatest lower bound for S_c. ▨

5. [10 points] Let S a non-empty subset of the real numbers that is bounded from below. Let c be a real number. Define the set

$$S_c = \{x + c \mid x \in S\}$$

Prove that $\inf(S_c) = \inf(S) + c$.

$\boxed{\text{Method 2}}$

Let $a = \inf(S)$, which exists since $S \neq \phi$ is bounded from below

$x \in S$.

① Then $a \leq x$ for all $x \in S$.
So, $a + c \leq x + c$ for all $x \in S$.
So, $a + c$ is a lower bound for S_c.

② Let $\varepsilon > 0$.
Since $a = \inf(S)$, there exists $x \in S$ such that $a \leq x < a + \varepsilon$.
So, $(a+c) \leq (x+c) < (a+c) + \varepsilon$.
What we have shown is that if $\varepsilon > 0$ then there exists $y \in S_c$ with $a+c \leq y < (a+c)+\varepsilon$.

By the useful sup/inf fact, $a + C = \inf(S_c)$,