**Test 2, Math 4650, Fall 2018** Name: Solutions

1. [12 points - 4 each]

(a) Fill in the definition: Let $S \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$. We say that $a$ is a limit point of $S$ if

for every $\varepsilon > 0$, there exists $x \in S$ with $0 < |x-a| < \varepsilon$.

$\boxed{\text{Note: } 0<|x-a|<\varepsilon \text{ takes care of both } x \neq a \text{ and } x \in (a-\varepsilon, a+\varepsilon).}$

(b) Fill in the definition: Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ and $a$ be a limit point of $D$. We say that $\lim_{x \to a} f(x) = L$ if

for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $x \in D$ and $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

(c) Fill in the definition: Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ and $a \in D$. We say that $f$ is continuous at $a$ if

for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $x \in D$ and $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

2. [10 points] Use the definition of limit to show that

$$\lim_{x \to 2} \frac{x-1}{x+2} = \frac{1}{4}$$

You must prove this straight from the definition to get credit, no theorems.

Let $\varepsilon > 0$.

Note that

$$\left| \frac{x-1}{x+2} - \frac{1}{4} \right| = \left| \frac{4x-4-x-2}{4(x+2)} \right| = \left| \frac{3x-6}{4(x+2)} \right| = \frac{3|x-2|}{4|x+2|}$$

Suppose $\delta \leq 1$.

Suppose $0 < |x-2| < \delta \leq 1$.

Then $-1 < x-2 < 1$.

So, $3 < x+2 < 5$.

Thus, $\frac{1}{3} > \frac{1}{x+2} > \frac{1}{5}$.

So, $\frac{1}{|x+2|} < \frac{1}{3}$,

Thus, if $0 < |x-2| \leq 1$, then $\left| \frac{x-1}{x+2} - \frac{1}{4} \right| = \frac{3|x-2|}{4|x+2|} < \frac{|x-2|}{4}$,

Let $\delta = \min\{1, 4\varepsilon\}$.

If $0 < |x-2| < \delta$, then

$$\left| \frac{x-1}{x+2} - \frac{1}{4} \right| < \frac{|x-2|}{4} < \frac{4\varepsilon}{4} = \varepsilon,$$

$\blacksquare$

3. [10 points] Use the definition of continuity to show that

$$f(x) = \frac{1}{\sqrt{x}}$$

is continuous for all $a > 0$

You must prove this straight from the definition to get credit, no theorems.

Let $\varepsilon > 0$.

Note that

$$\left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}\right| = \left|\frac{\sqrt{a}-\sqrt{x}}{\sqrt{a}\sqrt{x}}\right|$$

$$= \left|\frac{(\sqrt{a}-\sqrt{x})}{\sqrt{a}\sqrt{x}}\frac{(\sqrt{a}+\sqrt{x})}{(\sqrt{a}+\sqrt{x}}\right| = \left|\frac{a-x}{\sqrt{a}\sqrt{x}(\sqrt{a}+\sqrt{x})}\right|$$

$$= \frac{|x-a|}{\sqrt{a}\sqrt{x}(\sqrt{x}+\sqrt{a})}$$

Suppose that $\delta \le \frac{a}{2}$.

Suppose that $|x-a| < \delta \le \frac{a}{2}$.

Then, $-\frac{a}{2} < x-a < \frac{a}{2}$. So, $\frac{a}{2} < x < \frac{3a}{2}$.

Thus, $\sqrt{\frac{a}{2}} < \sqrt{x}$ and $\sqrt{\frac{a}{2}} + \sqrt{a} < \sqrt{x} + \sqrt{a}$.

So, $\frac{1}{\sqrt{x}} < \frac{1}{\sqrt{\frac{a}{2}}}$ and $\frac{1}{\sqrt{x}+\sqrt{a}} < \frac{1}{\sqrt{\frac{a}{2}}+\sqrt{a}}$.

So, if $|x-a| < \delta \le \frac{a}{2}$ then

$$\left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}\right| = \frac{|x-a|}{\sqrt{a}\sqrt{x}(\sqrt{x}+\sqrt{a})} < \frac{|x-a|}{\sqrt{a}\sqrt{\frac{a}{2}}(\sqrt{\frac{a}{2}}+\sqrt{a})}$$

Let $\delta = \min\left\{\frac{a}{2}, \varepsilon \cdot \sqrt{a} \cdot \sqrt{\frac{a}{2}} \cdot \left(\sqrt{\frac{a}{2}} + \sqrt{a}\right)\right\}$.

If $|x-a| < \delta$ then

$$\left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}\right| = \frac{|x-a|}{\sqrt{a}\sqrt{x}(\sqrt{x}+\sqrt{a})} < \frac{|x-a|}{\sqrt{a}\sqrt{\frac{a}{2}}(\sqrt{\frac{a}{2}}+\sqrt{a})} < \frac{\varepsilon\sqrt{a}\sqrt{\frac{a}{2}}(\sqrt{\frac{a}{2}}+\sqrt{a})}{\sqrt{a}\sqrt{\frac{a}{2}}(\sqrt{\frac{a}{2}}+\sqrt{a})}$$

$$= \varepsilon.$$

4. [10 points] Suppose that $\lim_{x \to \infty} f(x)$ exists and is equal to a real number $L$. Show that if $(a_n)$ is any unbounded increasing sequence of real numbers, then the sequence $(f(a_n))$ converges to $L$.

HW 3 #4(a)

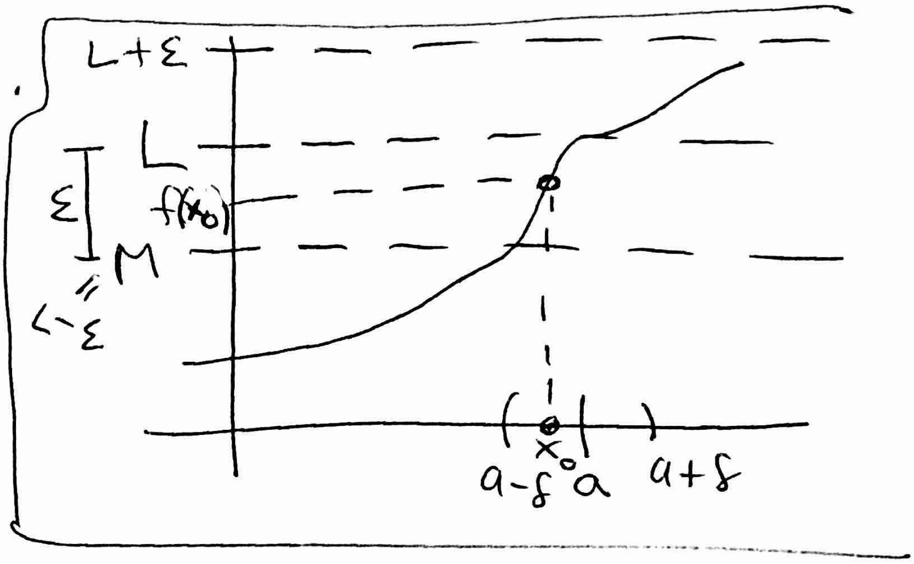
5. [10 points] Suppose that $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ and $a$ is a limit point of $D$. Suppose that there exists $M > 0$ such that $|f(x)| < M$ for all $x \in D$. Suppose further that $\lim_{x \to a} f(x) = L$ for some $L \in \mathbb{R}$. Prove that $L \leq M$.

Suppose that $L > M$.

We show this leads to a contradiction.

Let $\varepsilon = L - M$.

Since $\lim_{x \to a} f(x) = L$ there exists $\delta > 0$ so that if $x \in D$ and $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon = L - M$.




That is, if $x \in D$ and $0 < |x - a| < \delta$ then $-(L - M) < f(x) - L < L - M$.

So, if $x \in D$ and $0 < |x - a| < \delta$ then $-L + M < f(x) - L$, or $M < f(x)$.

Since $a$ is a limit point of $D$ there exists $x_0 \in D$ with $0 < |x_0 - a| < \delta$, and thus $f(x_0) > M$. This is a contradiction. $\blacksquare$