

Winter 2016 Math 465 Test 1 Take 1

Each question except (5) is worth 1000 points. Maximum possible score is 6000. Closed book, closed notes. No calculators. Show your work. No loitering, no smoking, no walking on the grass, no feeding the animals, no parking, no stopping or standing, no dumping, no running, no fishing, no vacancy.

1. Compute the inf, sup, max, and min (when they exist) of the following sets.

(a) $\{x \in \mathbb{Q} \mid x^2 < 2\}$

(b) $\{1 + \frac{1}{n} \mid n \in \mathbb{N}\}$

2. Find a number M such that $|3x^2 - 4x + 7| \leq M$ whenever $-2 \leq x \leq 4$. Show work to indicate how you got your answer and why it is correct.

3. Find an open cover of the interval $[0, 10)$ with no finite subcover.

4. Define the sequence $\{x_n\}$ by $x_n = \frac{1}{n} + (-1)^n$.

(a) Is $\{x_n\}$ bounded? Why or why not?

(b) Is $\{x_n\}$ monotone? Why or why not?

(c) Find a convergent subsequence of $\{x_n\}$. (You do not need to prove that your answer is correct.)

5. Please make sure the ringer on your phone is off.

Do TWO of the following three problems. Please indicate clearly which two you want me to grade. If you do more than two, I'll grade only the first two.

6. Let A and B be bounded sets such that $\emptyset \neq B \subset A \subset \mathbb{R}$. Prove that

$$\sup B \leq \sup A$$

7. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences such that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$. Define the sequence $\{x_n\}$ by

$$x_n = \begin{cases} a_{\lfloor (n+1)/2 \rfloor} & \text{if } n \text{ is odd} \\ b_{\lfloor n/2 \rfloor} & \text{if } n \text{ is even} \end{cases}$$

So $x_1 = a_1, x_2 = b_1, x_3 = a_2, x_4 = b_2, x_5 = a_3, x_6 = b_3$, and so on.

Prove that $\{x_n\}$ is convergent.

8. Let D be a nonempty set. Suppose that $f : D \rightarrow \mathbb{R}$ and $g : D \rightarrow \mathbb{R}$ are functions. Also suppose that $f(x) > 0$ and $g(x) > 0$ for all $x \in \mathbb{R}$. Prove that

$$\sup_{x \in D} [f(x)g(x)] \leq \left(\sup_{x \in D} f(x) \right) \left(\sup_{x \in D} g(x) \right)$$

EXTRA SPECIAL BONUS ITEM THAT IS NOT WORTH ANY POINTS: The word “planet” derives from the Greek word “wanderer,” so named because the planets appear to move against the background of fixed stars. There is a sea creature whose name also derives from this same Greek word. What is it?

Winter 2016 Math 465 Test 1 Take 1 solutions

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1. Compute the inf, sup, max, and min (when they exist) of the following sets.

(a) $\{x \in \mathbb{Q} \mid x^2 < 2\}$

Call this set A .

Then $\inf A = -\sqrt{2}$, $\sup A = \sqrt{2}$, and A has no max and no min.

(b) $\{1 + \frac{1}{n} \mid n \in \mathbb{N}\}$

Call this set B .

Then $\inf B = 1$, $\sup B = \max B = 2$, and B has no min.

2. Find a number M such that $|3x^2 - 4x + 7| \leq M$ whenever $-2 \leq x \leq 4$. Show work to indicate how you got your answer and why it is correct.

Let $M = 71$.

This works because:

$$\begin{aligned} |3x^2 - 4x + 7| &\leq |3x^2| + |4x| + |7| \text{ (by triangle inequality)} \\ &= 3|x|^2 + 4|x| + 7 \\ &\leq 3(4)^2 + 4(4) + 7 \text{ (because } -2 \leq x \leq 4) \\ &= 71. \end{aligned}$$

3. Find an open cover of the interval $[0, 10)$ with no finite subcover.

This was a homework problem with the answer provided.

4. Define the sequence $\{x_n\}$ by $x_n = \frac{1}{n} + (-1)^n$.

(a) Is $\{x_n\}$ bounded? Why or why not?

Yes, it is bounded, because $|x_n| \leq 2$ for all n .

(b) Is $\{x_n\}$ monotone? Why or why not?

No, it is not monotone, because the first three terms are $0, 3/2, -2/3$. The sequence increases from the first term to the second, but then decreases from the second to the third.

(c) Find a convergent subsequence of $\{x_n\}$. (You do not need to prove that your answer is correct.)

$\{x_{2n}\}$ is a convergent subsequence.

Specifically, it is the sequence $\frac{1}{n} + 1$, which converges to 1.

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Do TWO of the following three problems. Please indicate clearly which two you want me to grade. If you do more than two, I'll grade only the first two.

6. Let A and B be bounded sets such that $\emptyset \neq B \subset A \subset \mathbb{R}$. Prove that

$$\sup B \leq \sup A$$

We know that $\sup A$ is an upper bound for A , by def. of sup.

So $\sup A$ is an upper bound for B , because $B \subset A$.

Therefore $\sup B \leq \sup A$, by def. of sup.

7. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences such that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$. Define the sequence $\{x_n\}$ by

$$x_n = \begin{cases} a_{\lfloor (n+1)/2 \rfloor} & \text{if } n \text{ is odd} \\ b_{\lfloor n/2 \rfloor} & \text{if } n \text{ is even} \end{cases}$$

So $x_1 = a_1, x_2 = b_1, x_3 = a_2, x_4 = b_2, x_5 = a_3, x_6 = b_3$, and so on.

Prove that $\{x_n\}$ is convergent.

Proof: Let $x = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$.

[We will show that $x = \lim_{n \rightarrow \infty} x_n$.]

Let $\epsilon > 0$.

[We will show that there exists M such that if $n \geq M$, then $|x - x_n| < \epsilon$.

(*) Because $x = \lim_{n \rightarrow \infty} a_n$, we know that there exists M_a such that if $n \geq M_a$, then $|x - a_n| < \epsilon$.

(**) Because $x = \lim_{n \rightarrow \infty} b_n$, we know that there exists M_b such that if $n \geq M_b$, then $|x - b_n| < \epsilon$.

Let $M = \max\{2M_b, 2M_a - 1\}$. (How in the world did I come up with this??? Read the rest of the proof, and then think about how you could have found this M by working backwards.)

[We will show that if $n \geq M$, then $|x - x_n| < \epsilon$.]

Suppose $n \geq M$.

[We will show that $|x - x_n| < \epsilon$.]

Case 1: n is odd.

We know $n \geq 2M_a - 1$.

So $(n + 1)/2 \geq M_a$.

Therefore $|x - x_n| = |x - a_{(n+1)/2}| < \epsilon$, by (*).

Case 2: n is even.

We know $n \geq 2M_b$.

So $n/2 \geq M_b$.

Therefore $|x - x_n| = |x - b_{n/2}| < \epsilon$, by (**).

8. Let D be a nonempty set. Suppose that $f : D \rightarrow \mathbb{R}$ and $g : D \rightarrow \mathbb{R}$ are functions. Also suppose that $f(x) > 0$ and $g(x) > 0$ for all $x \in \mathbb{R}$. Prove that

$$\sup_{x \in D} [f(x)g(x)] \leq \left(\sup_{x \in D} f(x) \right) \left(\sup_{x \in D} g(x) \right)$$

Proof: Let $A = \sup \{f(x)g(x) \mid x \in D\}$.

We have that $\sup_{x \in D} [f(x)g(x)] = \sup A$.

Let $c = \sup_{x \in D} f(x)$ and let $d = \sup_{x \in D} g(x)$.

[By def. of sup, it suffices to show that cd is an upper bound for A .]

Let $f(x)g(x) \in A$ for some $x \in D$.

[We will show that $f(x)g(x) \leq cd$.]

By def. of sup, we know that c is an upper bound for $\{f(x) \mid x \in D\}$.

So $f(x) \leq c$.

By def. of sup, we know that d is an upper bound for $\{g(x) \mid x \in D\}$.

So $g(x) \leq d$.

Multiply these two inequalities to get $f(x)g(x) \leq cd$. (Here's where we use the fact that $f(x)$ and $g(x)$ are both positive. This guarantees that the inequality sign does not reverse when we multiply.)

Therefore cd is an upper bound for A , by def. of upper bound.

Therefore $\sup A \leq cd$, by def. of sup. (Behold! That's exactly what we needed to prove.)

EXTRA SPECIAL BONUS ITEM THAT IS NOT WORTH ANY POINTS: The word “planet” derives from the Greek word “wanderer,” so named because the planets appear to move against the background of fixed stars. There is a sea creature whose name also derives from this same Greek word. What is it?

Plankton

Winter 2016 Math 465 Test 1 Take 2

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1. Let $f(x) = x^2$, and let D be the closed interval $[-4, 2]$. Find $\sup_{x \in D} f(x)$ and $\inf_{x \in D} f(x)$.
2. Is the sequence $\{\frac{n}{n+1}\}$ convergent? If so, what is the limit?
3. Compute the inf, sup, min, and max (when they exist) of

$$\left\{ \frac{1}{m} - \frac{1}{n} \mid n, m \in \mathbb{N} \right\}.$$

4. Find a real number M such that $|x^5 - 2x^2 + 6x| \leq M$ for all x such that $-2 \leq x \leq 1$.
5. Please make sure the ringer on your phone is off.

Do TWO of the following three problems. Please indicate clearly which two you want me to grade. If you do more than two, I'll grade only the first two.

6. Let $x_n = \frac{n - \cos n}{n}$. Use the squeeze lemma to prove that $\{x_n\}$ converges, and find the limit.
7. Let A be a nonempty, bounded subset of \mathbb{R} such that $a > 0$ for all $a \in A$. Let $B = \{a^2 \mid a \in A\}$. Prove that $\sup B = (\sup A)^2$.
8. Prove that the union of two closed intervals is compact.

EXTRA SPECIAL BONUS ITEM THAT IS NOT WORTH ANY POINTS: Between the NFL, the NHL, the NBA, and MLB, name all teams whose names do not end in the letter S.

Winter 2016 Math 465 Test 1 Take 2 solutions

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1. Let $f(x) = x^2$, and let D be the closed interval $[-4, 2]$. Find $\sup_{x \in D} f(x)$ and $\inf_{x \in D} f(x)$. (You do not need to prove that your answer is correct, but as with all questions, you should show work.)

$$\sup_{x \in D} f(x) = 16$$

$$\inf_{x \in D} f(x) = 0$$

(Think about the graph of f .)

2. Is the sequence $\{\frac{n}{n+1}\}$ convergent? If so, what is the limit?

$$\text{We can write } \frac{n}{n+1} = \frac{1}{\frac{n+1}{n}} = \frac{1}{1+\frac{1}{n}}.$$

Now, $1 + 1/n$ is monotone decreasing, so $\frac{1}{1+\frac{1}{n}}$ is monotone increasing.

Also, $1 + 1/n \leq 2$ for all $n \in \mathbb{N}$, so $1/2 \leq \frac{1}{1+\frac{1}{n}} \leq 1$ for all $n \in \mathbb{N}$.

So this sequence is monotone and bounded. Therefore it converges.

$$\text{By Prop. 2.2.5, we have } \lim_{n \rightarrow \infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+\lim_{n \rightarrow \infty} \frac{1}{n}} = 1.$$

3. Compute the inf, sup, min, and max (when they exist) of

$$\left\{ \frac{1}{m} - \frac{1}{n} \mid n, m \in \mathbb{N} \right\}.$$

$$\text{Let } A = \left\{ \frac{1}{m} - \frac{1}{n} \mid n, m \in \mathbb{N} \right\}.$$

Then $\inf A = -1$ and $\sup A = 1$ and the min and max do not exist, because $-1 \notin A$ and $1 \notin A$.

4. Find a real number M such that $|x^5 - 2x^2 + 6x| \leq M$ for all x such that $-2 \leq x \leq 1$.

We have $|x^5 - 2x^2 + 6x| \leq |x^5| + |2x^2| + |6x| = |x|^5 + 2|x|^2 + 6|x|$ by triangle inequality.

Now, $|x|$ attains its maximum when $x = -2$ for $-2 \leq x \leq 1$.

So $M = |(-2)^5 + 2| - 2|^2 + 6| - 2| = 52$ will work.

5. Please make sure the ringer on your phone is off.

Do TWO of the following three problems. Please indicate clearly which two you want me to grade. If you do more than two, I'll grade only the first two.

6. Let $x_n = \frac{n - \cos n}{n}$. Use the squeeze lemma to prove that $\{x_n\}$ converges, and find the limit.

We have $\frac{n-1}{n} \leq x_n \leq \frac{n+1}{n}$.

So $1 - \frac{1}{n} \leq x_n \leq 1 + \frac{1}{n}$.

But $\lim_{n \rightarrow \infty} (1 - 1/n) = \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} 1/n = 1 - 0 = 1$.

Similarly, $\lim_{n \rightarrow \infty} (1 + 1/n) = 1$.

By the squeeze lemma, $\lim_{n \rightarrow \infty} x_n = 1$.

7. Let A be a nonempty, bounded subset of \mathbb{R} such that $a > 0$ for all $a \in A$. Let $B = \{a^2 \mid a \in A\}$. Prove that $\sup B = (\sup A)^2$.

Let $x = \sup A$.

[We will show that $x^2 = \sup B$.]

[First, we will show that x^2 is an upper bound for B .]

Let $b \in B$. [We will show that $b \leq x^2$.]

We know that $b = a^2$ for some $a \in A$, by def. of sup.

We know that $a \leq x$, because x is an upper bound for A , by def. of sup.

So $b = a^2 \leq x^2$. (The inequality does not reverse, because $a > 0$.)

[Next, we will show that if y is an upper bound for B , then $x^2 \leq y$.]

Suppose y is an upper bound for B .

Note that $y > 0$, because $y \geq a^2 > 0$ for all $a \in A$.

[We will show that \sqrt{y} is an upper bound for A .]

Let $a \in A$. [We will show that $a \leq \sqrt{y}$.]

We have $a^2 \in B$, by def. of B .

So $a^2 \leq y$, because y is an upper bound for B .

So $a \leq \sqrt{y}$. (Both sides are positive, so taking square root is legitimate.)

Therefore, \sqrt{y} is an upper bound for A .

So $x \leq \sqrt{y}$, by def. of sup. (Recall that $x = \sup A$.)

So $x^2 \leq y$.

Therefore, $x^2 = \sup B$, by def. of sup.

8. Prove that the union of two closed intervals is compact.

Let A and B be two closed intervals. Let $S = A \cup B$.

[We will show that S is compact.]

Let \mathcal{U} be an open cover of S .

[We will show that C has a finite subcover from \mathcal{U} .]

Then \mathcal{U} is an open cover of A , because $A \subset S$.

Now, A is compact, by the Heine-Borel theorem.

So A has a finite subcover from \mathcal{U} , by def. of compact.

In other words, there exist finitely many sets $C_1, \dots, C_n \in \mathcal{U}$ such that

$$A \subset C_1 \cup \dots \cup C_n.$$

Similarly, there exist finitely many sets $D_1, \dots, D_m \in \mathcal{U}$ such that

$$B \subset D_1 \cup \dots \cup D_m.$$

But then $C_1, \dots, C_n, D_1, \dots, D_m$ is a finite subcover of S from \mathcal{U} .

EXTRA SPECIAL BONUS ITEM THAT IS NOT WORTH ANY POINTS: Between the NFL, the NHL, the NBA, and MLB, name all teams whose names do not end in the letter S.

Oklahoma City Thunder, Miami Heat, Orlando Magic, Utah Jazz, Boston Red Sox, Chicago White Sox, Colorado Avalanche, Tampa Bay Lightning, Minnesota Wild

Winter 2016 Math 465 Test 2 Take 1

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1. Give an example of a sequence $\{x_n\}$ such that $\limsup_{n \rightarrow \infty} x_n = 5$ and $\liminf_{n \rightarrow \infty} x_n = -2$.

2. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

3. Let $c \in \mathbb{R}$. Find $\lim_{x \rightarrow c} (x^2 + x + 1)$.

4. What theorem guarantees that there exists a positive real number x such that $x^4 = 2$? Explain.

5. Please make sure the ringer on your phone is off.

Do TWO of the following three problems. Please indicate clearly which two you want me to grade. If you do more than two, I'll grade only the first two.

6. Prove that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 5x - 3$ is continuous by using the following definitions. We say $f : A \rightarrow \mathbb{R}$ is continuous if f is continuous at c for all $c \in A$. We say that f is continuous at c if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

7. Define the function f by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}.$$

Show that f is continuous at 1 and that f is not continuous at 2.

8. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = \sin x$. Prove that f is uniformly continuous. You may use the following facts from trigonometry, as well as any others you remember:

$$|\sin x| \leq |x| \text{ for all } x \in \mathbb{R}$$

$$|\cos x| \leq 1 \text{ for all } x \in \mathbb{R}$$

$$|\sin x| \leq 1 \text{ for all } x \in \mathbb{R}$$

$$\sin x - \sin y = 2 \sin((x - y)/2) \cos((x + y)/2) \text{ for all } x, y \in \mathbb{R}$$

EXTRA SPECIAL BONUS ITEM THAT IS NOT WORTH ANY POINTS: Name the California state bird; state flower; state animal; state vegetable; state fruit; and state historical society.

Winter 2016 Math 465 Test 2 Take 1 Solutions

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1. Give an example of a sequence $\{x_n\}$ such that $\limsup_{n \rightarrow \infty} x_n = 5$ and $\liminf_{n \rightarrow \infty} x_n = -2$.

Define $x_n = 5$ if n is odd and $x_n = -2$ if n is even.

2. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

Yes, it does, because $\frac{1}{n(n+1)} \leq 1/n^2$, and $\sum 1/n^2$ converges by p -test, so $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges by Comparison Test.

3. Let $c \in \mathbb{R}$. Find $\lim_{x \rightarrow c} (x^2 + x + 1)$.

The answer is $c^2 + c + 1$. This holds because the function $f(x) = x^2 + x + 1$ is a polynomial and is therefore continuous.

4. What theorem guarantees that there exists a positive real number x such that $x^4 = 2$? Explain.

Answer: The Intermediate Value Theorem. Let $f(x) = x^4$. Then f is a polynomial and is therefore continuous. Also, $f(0) = 0$ and $f(2) = 16$ and $0 < 2 < 16$. [Remark: This shows that $\sqrt[4]{2}$ exists. A similar argument shows that $\sqrt[d]{y}$ exists for every natural number d and every positive real number y .]

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Do TWO of the following three problems. Please indicate clearly which two you want me to grade. If you do more than two, I'll grade only the first two.

6. Prove that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 5x - 3$ is continuous by using the following definitions. We say $f : A \rightarrow \mathbb{R}$ is continuous if f is continuous at c for all $c \in A$. We say that f is continuous at c if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Proof: Let $c \in \mathbb{R}$. We will show that f is continuous at c .

Let $\epsilon > 0$.

[We will show that there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.]

Let $\delta = \epsilon/5$.

Suppose $|x - c| < \delta$. We will show that $|f(x) - f(c)| < \epsilon$.

Multiply both sides of $|x - c| < \delta$ by 5 to get $|5x - 5c| < 5\delta = \epsilon$.

Then $|f(x) - f(c)| = |(5x - 3) + (5c - 3)| = |5x - 5c| < \epsilon$.

7. Define the function f by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}.$$

Show that f is continuous at 1 and that f is not continuous at 2. (HINT: Use sequential limits).

Let $\{x_n\}$ be any sequence converging to 1. We will show that $\{f(x_n)\}$ converges to $1 = f(1)$, and hence f is continuous at 1.

If $\{x_n\}$ has infinitely many rational terms and infinitely many irrational terms, we create two subsequences. Let $\{x_{n_i}\}$ be the subsequence consisting of all rational terms in $\{x_n\}$ and let $\{x_{m_j}\}$ be the subsequence consisting of all irrational terms. Then

$$\lim_{i \rightarrow \infty} f(x_{n_i}) = \lim_{i \rightarrow \infty} x_{n_i} = 1$$

and

$$\lim_{i \rightarrow \infty} f(x_{m_i}) = \lim_{i \rightarrow \infty} x_{m_i}^2 = 1^2 = 1.$$

Given any $\epsilon > 0$ there exists M_1 and M_2 such that if $n_i \geq M_1$ then $|f(x_{n_i}) - 1| < \epsilon$ and if $m_i \geq M_2$ then $|f(x_{m_j}) - 1| < \epsilon$. Let $M = \max\{M_1, M_2\}$. Then if $n \geq M$ we have $|f(x_n) - 1| < \epsilon$. Hence $\{f(x_n)\}$ converges to $f(1)$.

If $\{x_n\}$ is a sequence with only finitely many rational terms, then there exists an $M \in \mathbb{N}$ such all x_n are irrational when $n \geq M$. It follows, as above that

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \geq M} f(x_n) = \lim_{n \rightarrow \infty} x_n^2 = 1.$$

Similarly, if $\{x_n\}$ has only finitely many irrational terms we get $\lim_{n \rightarrow \infty} f(x_n) = 1$. Therefore f is continuous at 1.

(i) **Using $\epsilon - \delta$ definition.** An alternate approach would be to use the definition of continuity directly. Given $\epsilon > 0$ we must find a $\delta > 0$ such that if $|x - 1| < \delta$ then $|f(x) - f(1)| < \epsilon$.

For x rational, we have that if $|x - 1| < \epsilon$ then

$$|f(x) - f(1)| < \epsilon,$$

since $f(x) = x$ and $f(1) = 1$.

For x irrational, it follows from our proof of the continuity of $g(x) = x^2$ that if

$$|x - 1| < \min\left\{\frac{\varepsilon}{3}, 1\right\}$$

then $|x^2 - 1| < \varepsilon$. Recall, this follows since if $|x - 1| < 1$ then $|x + 1| < 3$, and hence if $|x - 1| < \min\{1, \varepsilon/3\}$ then

$$\begin{aligned} |x^2 - 1| &= |x + 1||x - 1| \\ &< 3|x - 1| && \text{since } |x + 1| < 3 \\ &< \varepsilon && \text{since } |x - 1| < \frac{\varepsilon}{3}. \end{aligned}$$

8. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = \sin x$. Prove that f is uniformly continuous. You may use the following facts from trigonometry:

$$|\sin x| \leq |x| \text{ for all } x \in \mathbb{R}$$

$$|\cos x| \leq 1 \text{ for all } x \in \mathbb{R}$$

$$|\sin x| \leq 1 \text{ for all } x \in \mathbb{R}$$

$$\sin x - \sin y = 2 \sin((x - y)/2) \cos((x + y)/2) \text{ for all } x, y \in \mathbb{R}$$

Proof: Let $\epsilon > 0$. Let $x, y \in \mathbb{R}$.

[We will show that there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|\sin x - \sin y| < \epsilon$.]

Let $\delta = \epsilon$. (Note that δ depends only on ϵ , not on x and y .)

Suppose $|x - y| < \delta$.

[We will show that $|\sin x - \sin y| < \epsilon$.]

Example 3.2.6 on p. 105 shows that $|\sin x - \sin y| < |x - y|$.

Therefore $|\sin x - \sin y| < |x - y| < \delta = \epsilon$.

EXTRA SPECIAL BONUS ITEM THAT IS NOT WORTH ANY POINTS: Name the California state bird; state flower; state animal; state vegetable; state fruit; and state historical society.

California quail; California poppy; California grizzly bear; artichoke; avocado; California State Historical Society

Winter 2016 Math 465 Test 2 Take 2

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1. Let $x_n = \frac{(n-1)(-1)^n}{n}$. Find $\limsup x_n$ and $\liminf x_n$.

2. Does $\sum_{n=1}^{\infty} \frac{4^n}{n!}$ converge? Why or why not?

3. Define a function $f : (0, 1] \rightarrow \mathbb{R}$ by:

$$f(x) = -2(x - 1) \text{ if } 1/2 \leq x \leq 1$$

$$f(x) = 4(x - 1/4) \text{ if } 1/4 \leq x \leq 1/2$$

$$f(x) = -8(x - 1/4) \text{ if } 1/8 \leq x \leq 1/4$$

$$f(x) = 16(x - 1/16) \text{ if } 1/16 \leq x \leq 1/8$$

\vdots

$$f(x) = -2^n \left(x - \frac{1}{2^{n-1}}\right) \text{ if } \frac{1}{2^n} \leq x \leq \frac{1}{2^{n-1}} \text{ for all positive odd integers } n$$

$$f(x) = 2^n \left(x - \frac{1}{2^n}\right) \text{ if } \frac{1}{2^n} \leq x \leq \frac{1}{2^{n-1}} \text{ for all positive even integers } n$$

For example, $1/4 \leq 1/3 \leq 1/2$, so $f(1/3) = 4(1/3 - 1/4) = 4(1/12) = 1/3$.

Does $\lim_{x \rightarrow 0} f(x)$ exist? Why or why not? Hint: Consider the values $f(1), f(1/2), f(1/4), f(1/8), \dots$

4. Define $h : \mathbb{R} \rightarrow \mathbb{R}$ by $h(x) = \sin(x^2)$. Is h continuous? Why or why not?

5. Please make sure the ringer on your phone is off.

Do TWO of the following three problems. Please indicate clearly which two you want me to grade. If you do more than two, I'll grade only the first two.

6. Prove that $\{(n^2 - 1)/n\}$ is Cauchy directly using the definition of Cauchy sequences.

7. Let f be as in Problem (3). Prove that f is not uniformly continuous.

8. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Suppose that $f(x) \neq 0$ for all $x \in \mathbb{R}$. Prove that there exists $b > 0$ such that $|f(x)| \geq b$ for all $x \in [0, 1]$.

EXTRA SPECIAL BONUS ITEM THAT IS NOT WORTH ANY POINTS: Name the decade in which the dance was popular: the moonwalk; the Macarena; the Charleston; the bunny hop