MATH 4650-01 Analysis I - Quiz 2 prep

Cal State LA - Spring 2020

1. Let $x, y \in \mathbb{R}$. If $0 < x < y$, then $0 < 1/y < 1/x$.

2. Let $x, y \in \mathbb{R}$. If $xy > 0$ then, either both x and y are positive, or both are negative.

3. Let $A \subset \mathbb{R}$ be a nonempty set that is bounded below. Then $\inf A$ exists.

4. Let $B \subset \mathbb{R}$ be bounded. Let $A \subset B$ be a nonempty subset. Suppose all the inf's and sup's exist. Show that
$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

5. Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$. Then A is bounded above, B is bounded below, and $\sup A \leq \inf B$.

6. Let D be a nonempty set. Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded functions.

   (a) Show
   $$\sup(f(x) + g(x))_{x \in D} \leq \sup f(x)_{x \in D} + \sup g(x)_{x \in D}$$
   and
   $$\inf(f(x) + g(x))_{x \in D} \leq \inf f(x)_{x \in D} + \inf g(x)_{x \in D}.$$

(b) Find examples where we obtain strict inequalities.

7. Let $S \subset \mathbb{R}$ be a nonempty bounded set. Then there exist monotone sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n, y_n \in S$ and

$$\sup S = \lim_{n\to\infty} x_n \text{ and } \inf S = \lim_{n\to\infty} y_n.$$

8. Determine whether the following sequences are convergent. If yes, find the limit.

   (a) $x_n = \frac{(-1)^n}{2n}$.

   (b) $x_n = 2^{-n}$.

   (c) $x_n = \frac{n}{n^2+1}$.

   (d) $x_n = \frac{2^n}{n!}$.

MATH 4650-01 Analysis I - Quiz 2
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FULL NAME: ______________________ SCORE: __________

1. (6 points) Let $x, y \in \mathbb{R}$. Show that if $xy > 0$ then, either both x and y are positive, or both are negative.

2. (6 points) **Prove** that the sequence given by $x_n = \frac{(-1)^n}{n^2}$ is convergent. (*Reminder*: write a proof, do not use techniques you learned in Calculus).

3. Let D be a nonempty set. Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded functions.

   (a) (4 points) Show $\sup(f(x) + g(x))_{x \in D} \leq \sup f(x)_{x \in D} + \sup g(x)_{x \in D}$.

   (b) (4 points) Find examples where we obtain strict inequalities.

MATH 4650-01 Analysis I - Practice Problems - Quiz 3 prep

CAL STATE LA - SPRING 2020

1. Find lim inf and lim sup for the following sequences:

   (a) $x_n := \frac{(-1)^n}{n}$

   (b) $x_n := \frac{(n-1)(-1)^n}{n}$

2. If $S \subseteq \mathbb{R}$ is a set, then $x \in \mathbb{R}$ is a cluster point if for every $\epsilon > 0$, the set $(x - \epsilon, x + \epsilon) \cap S \setminus \{x\}$ is not empty. That is, if there are points of S arbitrarily close to x. For example, $S := \{1/n : n \in \mathbb{N}\}$ has a unique (only one) cluster point 0, but $0 \notin S$. Prove the following version of the Bolzano – Weierstrass theorem:

   Theorem. Let $S \subseteq \mathbb{R}$ be a bounded infinite set, then there exists at least one cluster point of S. Hint: If S is infinite, then S contains a countably infinite subset. That is, there is a sequence $\{x_n\}$ of distinct numbers in S.

3. Show that the following sequences are divergent:

   (a) $x_n := 1 - (-1)^n + \frac{1}{n}$

   (b) $x_n := \sin\left(\frac{n\pi}{4}\right)$

4. Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$ and that $\lim_{n\to\infty} (-1)^n x_n$ exists. Show that $\{x_n\}$ converges.

5. Show directly from the definition that the following are Cauchy sequences:

(a) $x_n := \frac{n+1}{n}$

(b) $x_n := \frac{n^2-1}{n^2}$

(c) $x_n := \left(1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$

6. Show directly from the definition that the following are not Cauchy sequences:

(a) $x_n := (-1)^n$

(b) $x_n := \left(n + \frac{(-1)^n}{n}\right)$

(c) $x_n := \ln n$

7. Show directly that a bounded, monotone increasing sequence is a Cauchy sequence.

8. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that $\{x_n\}$ is a Cauchy sequence.

9. Let $\{x_n\}$ and $\{y_n\}$ be sequences such that $\lim_{n\to\infty} y_n = 0$. Suppose that for all $k \in \mathbb{N}$ and for all $m \geq k$ we have

$$|x_m - x_k| < y_k.$$

Show that $\{x_n\}$ is a Cauchy sequence.

10. Decide the convergence or divergence of the following series.

(a) $\sum_{n=1}^{\infty} \frac{3}{9n+1}$

(b) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

(d) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(e) $\sum_{n=1}^{\infty} ne^{-n^2}$

MATH 4650-01 Analysis I - Quiz 4

Cal State LA - Spring 2020

Full Name: ______________________________ Score: ____________

Total: 20 points

1. (4 points) Let c_1 be a cluster point of $A \subseteq \mathbb{R}$ and c_2 be a cluster point of $B \subseteq \mathbb{R}$. Suppose $f : A \to B$ and $g : B \to C$ are functions such that

   (i) $f(x) \to c_2$ as $x \to c_1$.

   (ii) $g(y) \to L$ as $y \to c_2$.

   If $c_2 \in B$ also suppose that $g(c_2) = L$. Let $h(x) := g(f(x))$ and show $h(x) \to L$ as $x \to c_1$.

   (*Hint*: $f(x)$ could equal c_2 for many $x \in A$.)

2. (4 points) Using the definition of continuity directly prove that f is continuous at 1 and discontinuous at 2.

$$f(x) = \begin{cases} x & \text{if } \ x \in \mathbb{Q} \\ x^2 & \text{if } \ x \notin \mathbb{Q} \end{cases}$$

3. (4 points) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that for all rational numbers r, $f(r) = g(r)$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

4. (4 points) Let $f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ and $g(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

Are these functions continuous? Prove your assertions.

5. (4 points) Let $S \subseteq \mathbb{R}$ be given. Suppose $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are continuous functions and define $p : S \to \mathbb{R}$ and $q : S \to \mathbb{R}$ by

$$p(x) := \max\{f(x), g(x)\} \quad \text{and} \quad q(x) := \min\{f(x), g(x)\}.$$

Prove that p and q are continuous.

MATH 4650-01 Analysis I - Quiz 4

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FULL NAME: ______________________ SCORE: __________

Total: 20 points

1. (4 points) Let c_1 be a cluster point of $A \subseteq \mathbb{R}$ and c_2 be a cluster point of $B \subseteq \mathbb{R}$. Suppose $f : A \to B$ and $g : B \to C$ are functions such that

   (i) $f(x) \to c_2$ as $x \to c_1$.

   (ii) $g(y) \to L$ as $y \to c_2$.

   If $c_2 \in B$ also suppose that $g(c_2) = L$. Let $h(x) := g(f(x))$ and show $h(x) \to L$ as $x \to c_1$.

   (*Hint*: $f(x)$ could equal c_2 for many $x \in A$.)

   Fix $\epsilon > 0$. By (ii), there exists $\delta_1 > 0$ such that

   $$\forall y \in B \setminus \{c_2\}, \quad |y - c_2| < \delta_1 \Rightarrow |g(y) - L| < \epsilon. \tag{1}$$

   By (i), there exists $\delta_2 > 0$ such that

   $$\forall x \in A \setminus \{c_1\}, \quad |x - c_1| < \delta_2 \Rightarrow |f(x) - c_2| < \delta_1. \tag{2}$$

   CASE 1: Assume $c_2 \notin B$. Then, if $x \in A \setminus \{c_1\}$ then $f(x) \in B \setminus \{c_2\}$. Therefore, by (1) and (2),

   $$\forall x \in A \setminus \{c_1\}, \quad |x - c_1| < \delta_2 \Rightarrow |f(x) - c_2| < \delta_1 \Rightarrow |g(f(x)) - L| < \epsilon.$$

   CASE 2: Assume $c_2 \in B$. By assumption, $g(c_2) = L$. Fix $x \in A \setminus \{c_1\}$ with $|x - c_1| < \delta_2$. If $f(x) \neq c_2$ then proceed as in CASE 1. So assume $f(x) = c_2$. Then, $g(f(x)) = g(c_2) = L$. Then,

   $$|g(f(x)) - L| = 0 < \epsilon.$$

   In any case, $\forall x \in A \setminus \{c_1\}, \quad |x - c_1| < \delta_2 \Rightarrow |g(f(x)) - L| < \epsilon$.Therefore, $h(x) \to L$ as $x \to c_1$.

2. (4 points) Using the definition of continuity directly prove that f is continuous at 1 and discontinuous at 2.

$$f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ \\ x^2 & \text{if} \quad x \notin \mathbb{Q} \end{cases}$$

Fix $\epsilon > 0$. Note that

$$|f(x) - 1| = \begin{cases} |x-1| & \text{if} \quad x \in \mathbb{Q} \\ \\ |x^2 - 1| & \text{if} \quad x \notin \mathbb{Q} \end{cases} < 2|x-1|$$

for x small enough (say, $|x-1| < 1$). So if $\delta < \min\{1, \epsilon/2\}$, then

$$|x-1| < \delta \to |f(x) - 1| < \epsilon.$$

Since $f(1) = 1$, we conclude that f is continuous at 1.

Now, let $A = \{x \in \mathbb{Q} : x < 2\}$ and $B = \{x \notin \mathbb{Q} : x < 2\}$. Note that $\sup A = 2 = \sup B$, so there exist sequences $\{x_n\}$ in A and $\{y_n\}$ in B such that

$$\lim_{n\to\infty} x_n = 2 = \lim_{n\to\infty} y_n.$$

Note that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = 2 = f(2)$$

but

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} (y_n)^2 = 4 \neq f(2).$$

Therefore, f is not continuous at 2.

3. (4 points) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that for all rational numbers r, $f(r) = g(r)$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

   Fix $\epsilon > 0$ and $x \in \mathbb{R}$. Note that

$$x = \sup\{r \in \mathbb{Q} : r < x\}.$$

   So there is a sequence $\{r_n\}$ in $\mathbb{Q}$ such that $r_n \to x$. Since both f and g are continuous, there exist M_1 and M_2 in $\mathbb{N}$ such that

$$n \geq M_1 \Rightarrow |f(r_n) - f(x)| < \epsilon/2, \ \& \ \ n \geq M_2 \Rightarrow |g(r_n) - g(x)| < \epsilon/2$$

   Note also that

$$|f(x)-g(x)| = |f(x)-f(r_n)+f(r_n)-g(x)| = |f(x)-f(r_n)+g(r_n)-g(x)| \ \ \text{(by hypothesis, } f(r_n) = g(r_n))$$
$$\leq |f(x) - f(r_n)| + |g(r_n) - g(x).|$$

   Take $M = \max\{M_1, M_2\}$. Then,

$$n \geq M \Rightarrow |f(x) - g(x)| \leq |f(x) - f(r_n)| + |g(r_n) - g(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

   So for arbitrary $\epsilon > 0$ we have $|f(x) - g(x)| < \epsilon$. Thus, $f(x) = g(x)$.

4. (4 points) Let $f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ and $g(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

Are these functions continuous? Prove your assertions.

$\lim_{x\to 0} f(x)$ does not exists. Therefore, f is not continuous at 0. On the other hand,

$$\lim_{x\to 0} g(x) = 0 = g(0).$$

So, g is continuous at 0. Fix $c \neq 0$. Note that $h(x) = 1/x$ is continuos at c and $h(c) \neq 0$. Therefore, we can use properties of limits and continuous functions to verify the continuity of g at c:

$$\lim_{x\to c} g(x) = \lim_{x\to c} x\sin(1/x) = \lim_{x\to c} x \cdot \lim_{x\to c} \sin(1/x) = c \cdot \sin\left(\lim_{x\to c} 1/x\right) = c \cdot \sin(1/c) = g(c).$$

Thus, g is continuous on $\mathbb{R}$.

5. (4 points) Let $S \subseteq \mathbb{R}$ be given. Suppose $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are continuous functions and define $p : S \to \mathbb{R}$ and $q : S \to \mathbb{R}$ by

$$p(x) := \max\{f(x), g(x)\} \quad \text{and} \quad q(x) := \min\{f(x), g(x)\}.$$

Prove that p and q are continuous.

Note that

$$p(x) := \max\{f(x), g(x)\} \;= \frac{1}{2}\,(f(x) + g(x)) + \frac{1}{2}|f(x) + g(x)|$$

and

$$q(x) := \min\{f(x), g(x)\} \;= \frac{1}{2}\,(f(x) + g(x)) - \frac{1}{2}|f(x) + g(x)|;$$

so the continuity of p and q follows from the these fact: addition and subtraction of continuous functions yield continuous functions, multiplying a continuous function by a constant yields a continuous function, and the absolute value function $h(x) = |x|$ (prove it!).

MATH 4650-01 Analysis I - Midterm 1 prep

Cal State LA - Spring 2020

1. (3.3.1) Find an example of a discontinuous function $f : [0, 1] \longrightarrow \mathbb{R}$ where the conclusion of the intermediate value theorem fails.

2. (3.3.3) Let $f : (0, 1) \longrightarrow \mathbb{R}$ be a continuous function such that $\lim_{x\to 0} f(x) = \lim_{x\to 1} f(x) = 0$. Show that f achieves either an absolute minimum or an absolute maximum on $(0, 1)$ (but perhaps not both).

3. (3.3.13) True/False, prove or find a counterexample. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function such that $f|_{\mathbb{Z}}$ is bounded, then f is bounded.

4. (3.3.15) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

   (a) Prove that if there is a c such that $f(c)f(-c) < 0$, then there is a $d \in \mathbb{R}$ such that $f(d) = 0$.

   (b) Find a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(\mathbb{R}) = \mathbb{R}$ (i.e., f is onto), but $f(x)f(-x) \geq 0$ for all $x \in \mathbb{R}$.

5. (3.4.3) Show that $f : (c, \infty) \longrightarrow \mathbb{R}$ for some $c > 0$ and be defined by $f(x) := 1/x$ is Lipschitz continuous.

6. (3.4.4) Show that $f : (0, \infty) \longrightarrow \mathbb{R}$ defined by $f(x) := 1/x$ is not Lipschitz continuous.

7. (3.4.8) Show that $f : (0, \infty) \longrightarrow \mathbb{R}$ defined by $f(x) := \sin(1/x)$ is not uniformly continuous.

8. (3.4.10)

   (a) Find a continuous $f : (0, 1) \longrightarrow \mathbb{R}$ and a sequence $\{x_n\}$ in $(0, 1)$that is Cauchy, but such that $\{f(x_n)\}$ is not Cauchy.

   (b) Prove that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, and $\{x_n\}$ is Cauchy, then $\{f(x_n)\}$ is Cauchy.

9. (3.4.11)

   (a) If $f : S \longrightarrow \mathbb{R}$ and $g : S \longrightarrow \mathbb{R}$ are uniformly continuous, then $h : S \longrightarrow \mathbb{R}$ given by $h(x) := f(x) + g(x)$ is uniformly continuous.

   (b) If $f : S \longrightarrow \mathbb{R}$ is uniformly continuous and $a \in \mathbb{R}$, then $h : S \longrightarrow \mathbb{R}$ given by $h(x) := af(x)$ is uniformly continuous.

10. (3.4.12)

    (a) If $f : S \longrightarrow \mathbb{R}$ and $g : S \longrightarrow \mathbb{R}$ are Lipschitz continuous, then $h : S \longrightarrow \mathbb{R}$ given by $h(x) := f(x) + g(x)$ is Lipschitz continuous.

    (b) If $f : S \longrightarrow \mathbb{R}$ is Lipschitz continuous and $a \in \mathbb{R}$, then $h : S \longrightarrow \mathbb{R}$ given by $h(x) := af(x)$ is Lipschitz continuous.

MATH 4650-01 Analysis I - Practice Problems - Quiz 3 prep

Cal State LA - Spring 2020

Full Name: ______________________________ Score: ____________

Total: 20 points

1. (4 points) Show that the following sequences are divergent:

   (a) $x_n := 1 - (-1)^n + \frac{1}{n}$

   (b) $x_n := \sin\left(\frac{n\pi}{4}\right)$

2. (4 points) Decide the convergence or divergence of the following series.

   (a) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

   (b) $\sum_{n=1}^{\infty} ne^{-n^2}$

3. (4 points) Show directly from the definition that the following are Cauchy sequences:

   (a) $x_n := \frac{n^3-1}{n^3}$

   (b) $x_n := \left(1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$

4. (4 points) If $x_n := \sqrt{n}$, show that $\{x_n\}$ satisfies $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$, but that it is not a Cauchy sequence.

5. (4 points) Let $\{x_n\}$ and $\{y_n\}$ be sequences such that $\lim_{n\to\infty} y_n = 0$. Suppose that for all $k \in \mathbb{N}$ and for all $m \geq k$ we have

$$|x_m - x_k| < y_k.$$

Show that $\{x_n\}$ is a Cauchy sequence.

MATH 4650-01 Analysis I - Midterm 1

Cal State LA - Spring 2020

Full Name: ______________________ Score: __________

Total: 25 points

1. (4 points) Prove by induction. For a finite set A of cardinality n, the cardinality of $\mathcal{P}(A)$ is 2^n.

2. (4 points) Let $x, y, z \in \mathbb{R}$. Prove the following.

   (a) If $0 < x < y$, then $0 < 1/y < 1/x$.

   (b) If $x \leq y$ and $z \leq w$,then $x + z \leq y + w$.

3. (4 points) Let $S \subset \mathbb{R}$ be a nonempty bounded set. Then there exist monotone sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n, y_n \in S$ and

$$\sup S = \lim_{n\to\infty} x_n \text{ and } \inf S = \lim_{n\to\infty} y_n.$$

4. (4 points) A sequence is $\{x_n\}$ is said to be **bounded** if it is bounded as a function, that is, there exists a number $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Show that every convergent sequence is bounded.

5. (6 points)
(a) Show that $x_n = \frac{n^2}{2^n}$ converges to 0.
(b) Use the Squeeze Lemma to show that the sequence

$$x_n = \frac{n^2 - \cos^2(n)}{2^n}$$

converges to 0. *Hint:* Part (a) is useful here.

6. (3 points) Use the definition of convergence to show that the sequence

$$x_n = \frac{(-1)^{n-1}\sqrt{n}}{n+1}$$

is convergent.

MATH 4650-01 Analysis I - Midterm 1 - Retake

Cal State LA - Spring 2020

Full Name: ______________________ Score: __________

Total: 25 points

1. (4 points) Prove by induction. Let $A \subset \mathbb{R}$ be a nonempty finite subset. Then A is bounded. Furthermore, both $\inf A$ and $\sup A$ exist and are in A .

2. (4 points) Mark the following statements True or False. Explain.

   (i) If $\{x_n\}$ is a sequence such that $\{x_n^2\}$ converges, then $\{x_n\}$ converges.

   (ii) If $\{a_n\}$ is a bounded sequence and $\{b_n\}$ is a sequence converging to 0, then $\{a_n b_n\}$ converges to 0.

3. (4 points) Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$. Then A is bounded above, B is bounded below, and $\sup A \leq \inf B$.

4. (4 points) A sequence is $\{x_n\}$ is said to be **bounded** if it is bounded as a function, that is, there exists a number $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Show that every convergent sequence is bounded.

5. (6 points)
(a)Use the definition of convergence to show that the sequence

$$x_n = \frac{\sqrt{n}}{n+1}$$

converges to 0.
(b) Use the Squeeze Lemma to show that the sequence

$$x_n = \frac{\sqrt{n} - \cos^2(n)}{n+1}$$

converges to 0. *Hint:* Part (a) is useful here.

6. (3 points) Use the Ratio Test to show that the sequence

$$x_n = \frac{2^n}{n!}$$

is convergent.

MATH 4650-01 Analysis I - Midterm 2 prep

Cal State LA - Spring 2020

1. (3.3.1) Find an example of a discontinuous function $f : [0,1] \longrightarrow \mathbb{R}$ where the conclusion of the intermediate value theorem fails.
2. (3.3.3) Let $f : (0,1) \longrightarrow \mathbb{R}$ be a continuous function such that $\lim_{x\to 0} f(x) = \lim_{x\to 1} f(x) = 0$. Show that f achieves either an absolute minimum or an absolute maximum on $(0,1)$ (but perhaps not both).
3. (3.3.13) True/False, prove or find a counterexample. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function such that $f|_{\mathbb{Z}}$ is bounded, then f is bounded.
4. (3.3.15) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.
   (a) Prove that if there is a c such that $f(c)f(-c) < 0$, then there is a $d \in \mathbb{R}$ such that $f(d) = 0$.
   (b) Find a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(\mathbb{R}) = \mathbb{R}$ (i.e., f is onto), but $f(x)f(-x) \geq 0$ for all $x \in \mathbb{R}$.
5. (3.4.3) Show that $f : (c,\infty) \longrightarrow \mathbb{R}$ for some $c > 0$ and be defined by $f(x) := 1/x$ is Lipschitz continuous.
6. (3.4.4) Show that $f : (0,\infty) \longrightarrow \mathbb{R}$ defined by $f(x) := 1/x$ is not Lipschitz continuous.
7. (3.4.8) Show that $f : (0,\infty) \longrightarrow \mathbb{R}$ defined by $f(x) := \sin(1/x)$ is not uniformly continuous.
8. (3.4.10)
   (a) Find a continuous $f : (0,1) \longrightarrow \mathbb{R}$ and a sequence $\{x_n\}$ in $(0,1)$that is Cauchy, but such that $\{f(x_n)\}$ is not Cauchy.
   (b) Prove that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, and $\{x_n\}$ is Cauchy, then $\{f(x_n)\}$ is Cauchy.
9. (3.4.11)
   (a) If $f : S \longrightarrow \mathbb{R}$ and $g : S \longrightarrow \mathbb{R}$ are uniformly continuous, then $h : S \longrightarrow \mathbb{R}$ given by $h(x) := f(x) + g(x)$ is uniformly continuous.
   (b) If $f : S \longrightarrow \mathbb{R}$ is uniformly continuous and $a \in \mathbb{R}$, then $h : S \longrightarrow \mathbb{R}$ given by $h(x) := af(x)$ is uniformly continuous.
10. (3.4.12)
   (a) If $f : S \longrightarrow \mathbb{R}$ and $g : S \longrightarrow \mathbb{R}$ are Lipschitz continuous, then $h : S \longrightarrow \mathbb{R}$ given by $h(x) := f(x) + g(x)$ is Lipschitz continuous.
   (b) If $f : S \longrightarrow \mathbb{R}$ is Lipschitz continuous and $a \in \mathbb{R}$, then $h : S \longrightarrow \mathbb{R}$ given by $h(x) := af(x)$ is Lipschitz continuous.

MATH 4650-01 Analysis I - Midterm 2 - printable version

Cal State LA - Spring 2020

Full Name: ______________________________ Score: ____________

Total: 25 points

1. (3 points) Find an example of a discontinuous function $f : [0, 1] \longrightarrow \mathbb{R}$ where the conclusion of the intermediate value theorem fails.

2. (6 points) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

a) Prove that if there is a c such that $f(c)f(-c) < 0$, then there is a $d \in \mathbb{R}$ such that $f(d) = 0$.

b) Find a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(\mathbb{R}) = \mathbb{R}$ (i.e., f is onto), but $f(x)f(-x) \geq 0$ for all $x \in \mathbb{R}$.

3. (4 points) Let $c \geq 0$ be given. Consider $f : (c, \infty) \longrightarrow \mathbb{R}$ defined $f(x) = 1/x$. Prove the following:

a) If $c > 0$ then, f is Lipschitz continuous.

b) If $c = 0$ then, f is not Lipschitz continuous.

4. (4 points) Let $f : (0, 1) \longrightarrow \mathbb{R}$ be a continuous function such that $\lim_{x \to 0} f(x) = \lim_{x \to 1} f(x) = 0$. Show that f achieves either an absolute minimum or an absolute maximum on $(0, 1)$ (but perhaps not both).

5. (4 points)

a) If $f : S \longrightarrow \mathbb{R}$ and $g : S \longrightarrow \mathbb{R}$ are uniformly continuous, then $h : S \longrightarrow \mathbb{R}$ given by $h(x) := f(x) + g(x)$ is uniformly continuous.

b) If $f : S \longrightarrow \mathbb{R}$ is uniformly continuous and $a \in \mathbb{R}$, then $h : S \longrightarrow \mathbb{R}$ given by $h(x) := af(x)$ is uniformly continuous.

6. (4 points) Suppose $f : S \longrightarrow \mathbb{R}$ and $g : [0, \infty) \longrightarrow [0, \infty)$ are functions, g is continuous at 0 and whenever x and y are in S we have $|f(x) - f(y)| \leq g(|x - y|)$. Prove that f is uniformly continuous.