**Def:** A function $f: \mathbb{C} \to \mathbb{C}$ is called <u>entire</u> if $f'(z)$ exists for all $z \in \mathbb{C}$. [or $f$ is analytic on all of $\mathbb{C}$]

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**Ex:** polynomials are entire functions

Ex: Let $f(z) = e^z$. Let's show that $f$ is entire.

$$f(x+iy) = e^{x+iy} = \underbrace{e^x \cos(y)}_{u(x,y)} + i\,\underbrace{e^x \sin(y)}_{v(x,y)}$$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos(y) \\ \frac{\partial v}{\partial y} &= e^x \cos(y) \\ \frac{\partial u}{\partial y} &= -e^x \sin(y) \\ \frac{\partial v}{\partial x} &= e^x \sin(y) \end{aligned}}$$

- $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous everywhere
- CR-equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are true everywhere

So, $f'(z)$ exists for all $z \in \mathbb{C}$ and

$$f'(z) = f'(x+iy) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = e^x \cos(y) + i e^x \sin(y)$$

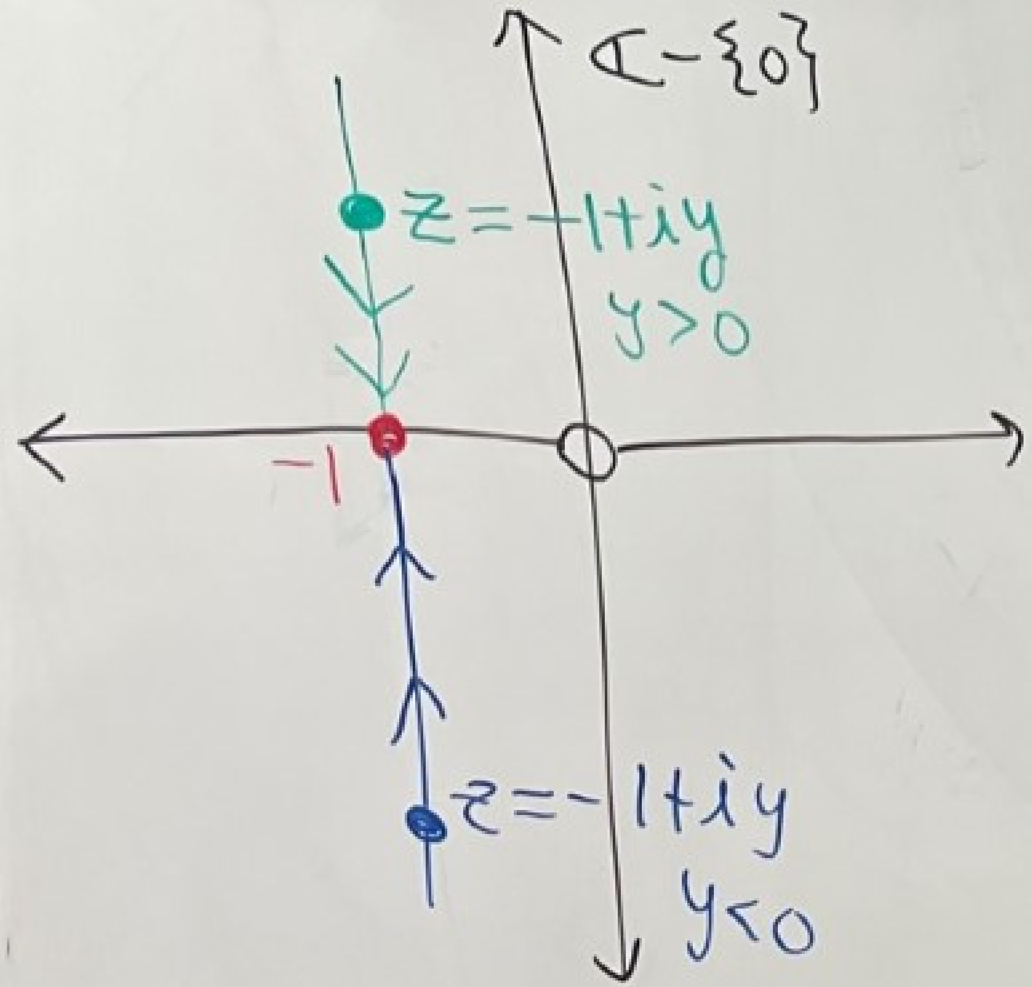
$$= e^x[\cos(y) + i\sin(y)] = e^z$$

So, $f$ is entire and $f'(z) = e^z$

Ex: (log has discontinuities at its "branch line")

Consider log with branch $[-\pi, \pi)$.

Here $\log : \mathbb{C} - \{0\} \longrightarrow \mathbb{C}$ and $\log(z) = \ln(|z|) + i\arg(z)$ where $-\pi \le \arg(z) < \pi$




If $z = -1 + iy$ where $y > 0$ then

$$\log(z) = \ln|z| + i\arg(z) \longrightarrow \overbrace{\ln|-1|}^{\ln(1)=0} + i\pi = i\pi$$

as $y \to 0$ or $z \to 1$ from above

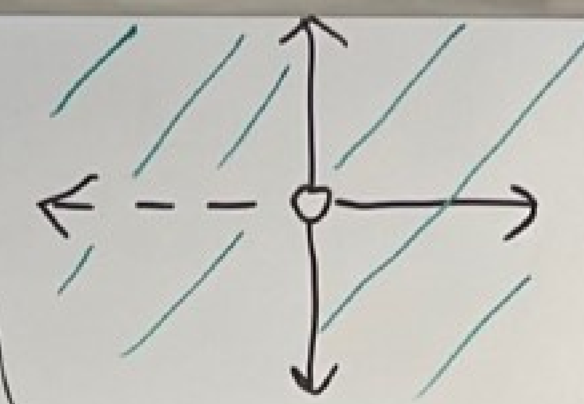
If $z = -1 + iy$ where $y < 0$ then

$$\log(z) = \ln|z| + i\arg(z) \longrightarrow \ln|-1| + i(-\pi) = -i\pi$$

So log with branch $[-\pi, \pi)$ is discontinuous at $z = -1$. It's discontinuous on the negative x-axis.

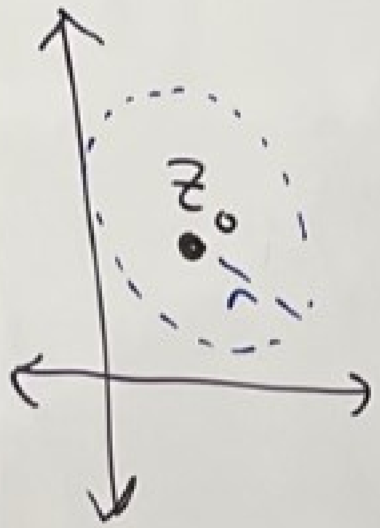
$y=0$

You can make log with branch $[-\pi,\pi)$ continuous if you restrict it to $\mathbb{C}-\{x+iy \mid x\le 0, \cancel{y\in\mathbb{R}}\}$



**Theorem** (Polar version of Cauchy-Riemann equations)

Let $f(z)=f(re^{i\theta})=u(r,\theta)+iv(r,\theta)$ be defined on some $D(z_0;r)$ where $z_0=r_0e^{i\theta_0}$.




Suppose $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ exist and are continuous on $D(z_0;r)$.

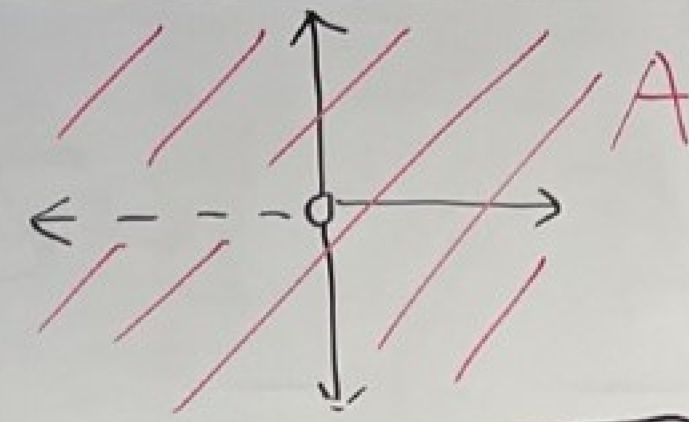
If $\boxed{\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta}=-\frac{\partial v}{\partial r}}$ at the point $(r_0,\theta_0)$, then $f'(z_0)$ exists

Cauchy-Riemann equations

and $$\boxed{f'(z_0)=e^{-i\theta_0}\left[\frac{\partial u}{\partial r}(r_0,\theta_0)+i\frac{\partial v}{\partial r}(r_0,\theta_0)\right]}$$

proof in Hoffmann/Marsden book

Ex: Let $A = \mathbb{C} - \{x+iy \mid x \le 0 \text{ and } y = 0\}$




Define $\log : A \to \mathbb{C}$ by $\log(z) = \ln|z| + i\arg(z)$ with $-\pi < \arg(z) < \pi$. This is called the principal branch of the logarithm

$\boxed{\text{Claim: This log is analytic on } A \text{ and } \frac{d}{dz}\log(z) = \frac{1}{z} \text{ for all } z \in A. \text{ (Similar statements are true for other branches of log)}}$

proof: $\log(z) = \log(re^{i\theta}) = \underbrace{\ln(r)}_{u(r,\theta)=\ln(r)} + i\underbrace{\theta}_{v(r,\theta)=\theta}$ where $-\pi < \theta < \pi$, $r > 0$.

$$\boxed{\begin{array}{l} \frac{\partial u}{\partial r} = \frac{1}{r} \\ \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\cdot 1 = \frac{1}{r} \end{array}}$$

$$\boxed{\begin{array}{l} \frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{1}{r}\cdot 0 = 0 \\ -\frac{\partial v}{\partial r} = -0 = 0 \end{array}}$$

- $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ exist and are continuous on $A$.
- CR-equations: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$ on $A$.

So, for all $z = re^{i\theta}$, $f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = e^{-i\theta}\left[\frac{1}{r}\right] = \frac{1}{re^{i\theta}}$

$$= \frac{1}{z}$$

**Claim:** $\sin(z)$ and $\cos(z)$ are entire functions.

And, $\frac{d}{dz}\sin(z) = \cos(z)$ and $\frac{d}{dz}\cos(z) = -\sin(z)$ for all $z \in \mathbb{C}$

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**proof:** Let $f(z) = \sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i} = \dfrac{1}{2i}\left[e^{iz} - e^{-iz}\right]$

$iz$ and $-iz$ are polynomials so are entire.

$e^{z}$ is entire.

So, $\underbrace{e^{iz} \text{ and } e^{-iz}}_{\text{composing entire functions}}$ are entire and hence $\underbrace{\frac{1}{2i}\left[e^{iz} - e^{-iz}\right]}_{\text{scaling/subtracting entire functions}}$ is entire by our theorems from class.

$$f'(z) \underset{\text{chain rule}}{=} \frac{1}{2i}\left[i e^{iz} - (-i)e^{-iz}\right] = \frac{1}{2}\left[e^{iz} + e^{-iz}\right] = \cos(z).$$

Try for $\cos(z)$ for practice $\blacksquare$

Ex: Let $a \in \mathbb{C}$ where $a \neq 0$.

Define $f: \mathbb{C} \to \mathbb{C}$ by $f(z) = a^z = e^{z \log(a)}$

where log is some branch of log

ie $\log(a) = \ln|a| + i\arg(a)$ where $c \leq \arg(a) < c + 2\pi$ for some $c$.

$\boxed{\text{Claim: } f \text{ is entire and } f'(z) = [\log(a)]\, a^z}$

proof: $\log(a)$ is a constant so we are plugging an entire function $\log(a) \cdot z$ into an entire function $e^z$ to get $f$. So, $f$ is entire.

And,

$$f'(z) = \left(e^{z \log(a)}\right)' = \log(a) \cdot e^{z \log(a)} = [\log(a)] \cdot a^z \qquad \blacksquare$$