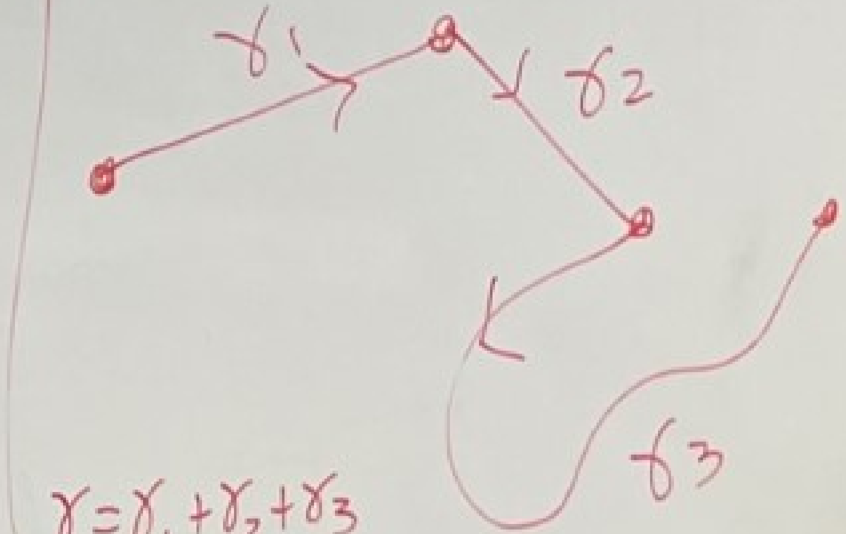
# Proof continued from last time $\left[ |f(z)| \le M \ \forall z \text{ on } \gamma \rightarrow \left|\int_\gamma f\right| \le M \cdot \text{arclength}(\gamma) \right]$

We proved the thm for smooth curves.

What if $\gamma$ is piece-wise smooth?

So, $\gamma = \sum_{i=1}^{n} \gamma_i$ where each $\gamma_i$ is smooth (the endpoint of $\gamma_i$ = starting point of $\gamma_{i+1}$)




$\gamma = \gamma_1 + \gamma_2 + \gamma_3$

pic for $n=3$

Then,

$$\left|\int_\gamma f\right| = \left|\sum_{i=1}^{n} \int_{\gamma_i} f\right| \overset{\Delta}{\le} \sum_{i=1}^{n} \left|\int_{\gamma_i} f\right|$$

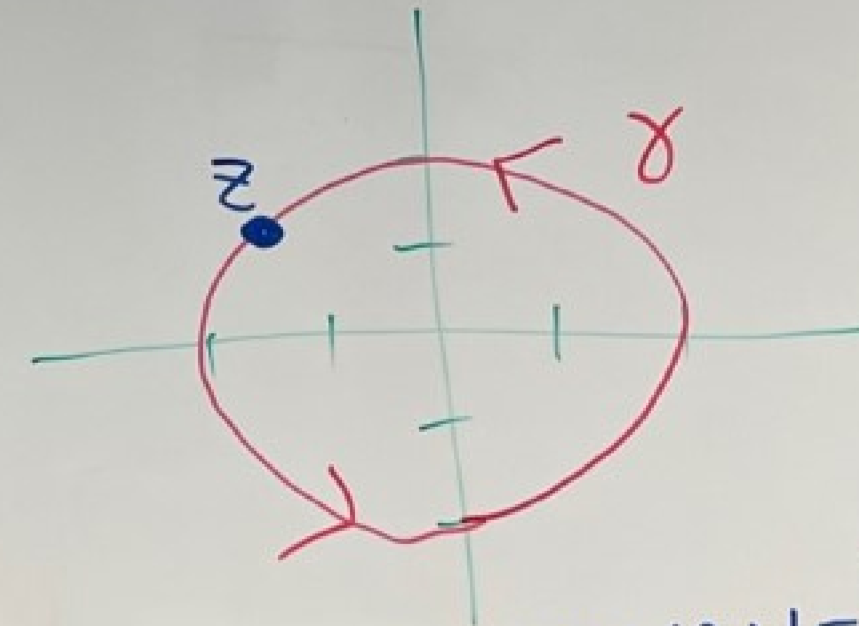
$$\overset{\text{last time}}{\le} \sum_{i=1}^{n} M \cdot \text{arclength}(\gamma_i)$$

$$= M \cdot \sum_{i=1}^{n} \text{arclength}(\gamma_i) = M \cdot \text{arclength}(\gamma). \quad \blacksquare$$

Ex: Let $f(z) = 3z^2 - 5z + 1$.

Let $\gamma$ be the circle centered at $0$ with radius $2$, oriented counter-clockwise.

Bound $\left|\int_\gamma f\right|$.




Let $z$ be on $\gamma$. Then, $|z| = 2$.

So if $z$ is on $\gamma$, then

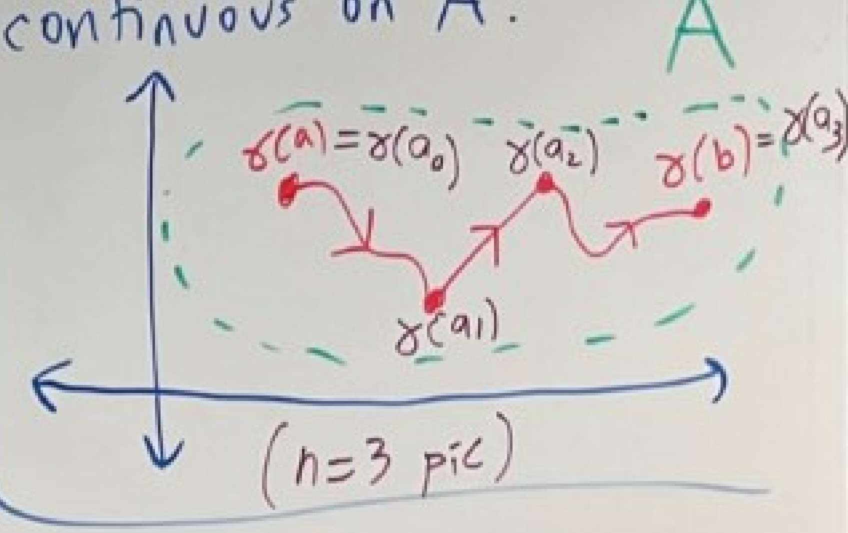
$$|f(z)| = |3z^2 - 5z + 1| \overset{\triangle}{\le} |3z^2| + |-5z| + |1|$$
$$= 3|z|^2 + 5|z| + 1 = 3\cdot 2^2 + 5\cdot 2 + 1 = 12 + 10 + 1 = \overset{M}{\boxed{23}}$$

So if $z$ is on $\gamma$, then $|f(z)| \le 23$.

Thus, $$\left|\int_\gamma f\right| \le 23 \cdot \text{arclength}(\gamma) = 23 \cdot \overbrace{(2\pi(2))}^{2\pi r} = \boxed{92\pi}$$

**Fundamental Theorem of Calculus** – Suppose that $\gamma : [a,b] \to \mathbb{C}$ is a piecewise smooth curve and $F$ is a function defined and analytic on an open set $A$ that contains $\gamma$. Assume $F'$ is continuous on $A$.

Then, $\int_\gamma F' = F(\gamma(b)) - F(\gamma(a))$




proof: Break $[a,b]$ into sub-intervals $[a_i, a_{i+1}]$ where $\gamma'$ exists on each $(a_i, a_{i+1})$ and is continuous on each $[a_i, a_{i+1}]$ where $a_0 = a$, $a_n = b$.

Suppose $F(\gamma(t)) = u(t) + i v(t)$. So, $F'(\gamma(t)) \cdot \gamma'(t) = u'(t) + i v'(t)$.

Then, $\int_\gamma F' = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} F'(\gamma(t)) \cdot \gamma'(t)\, dt = \dots$

formula for complex integration

$$\int_\gamma F' = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} F'(\gamma(t)) \cdot \gamma'(t)\,dt = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} \left[u'(t) + i v'(t)\right] dt$$

$$= \sum_{i=0}^{n-1} \left[\int_{a_i}^{a_{i+1}} u'(t)\,dt + i \int_{a_i}^{a_{i+1}} v'(t)\,dt\right] = \sum_{i=0}^{n-1} \left[\left(u(a_{i+1}) - u(a_i)\right) + i\left(v(a_{i+1}) - v(a_i)\right)\right]$$

FTOC from calculus/real analysis

$$= \underbrace{\left[\left(u(a_1) - u(a_0)\right) + i\left(v(a_1) - v(a_0)\right)\right]}_{(i=0 \text{ term})} + \underbrace{\left[\left(u(a_2) - u(a_1)\right) + i\left(v(a_2) - v(a_1)\right)\right]}_{(i=1 \text{ term})} + \underbrace{\left[\left(u(a_3) - u(a_2)\right) + i\left(v(a_3) - v(a_2)\right)\right]}_{(i=2 \text{ term})}$$

$$+ \dots + \underbrace{\left[\left(u(a_{n-1}) - u(a_{n-2})\right) + i\left(v(a_{n-1}) - v(a_{n-2})\right)\right]}_{(i=n-2 \text{ term})} + \underbrace{\left[\left(u(a_n) - u(a_{n-1})\right) + i\left(v(a_n) - v(a_{n-1})\right)\right]}_{(i=n-1 \text{ term})}$$

$$= -u(a_0) + u(a_n) + i\left(-v(a_0) + v(a_n)\right)$$

$$= \left(u(a_n) + i v(a_n)\right) - \left(u(a_0) + i v(a_0)\right) = F(\gamma(a_n)) - F(\gamma(a_0)) = F(\gamma(b)) - F(\gamma(a)).$$

$\blacksquare$

<u>Theorem:</u> Suppose that $A \subseteq \mathbb{C}$ and $A$ is a region (open and path-connected) and $f : A \to \mathbb{C}$ is analytic on $A$ and $f'(z) = 0$ for all $z \in A$. Then, there exists $c \in \mathbb{C}$ where $f(z) = c$ for all $z \in A$.

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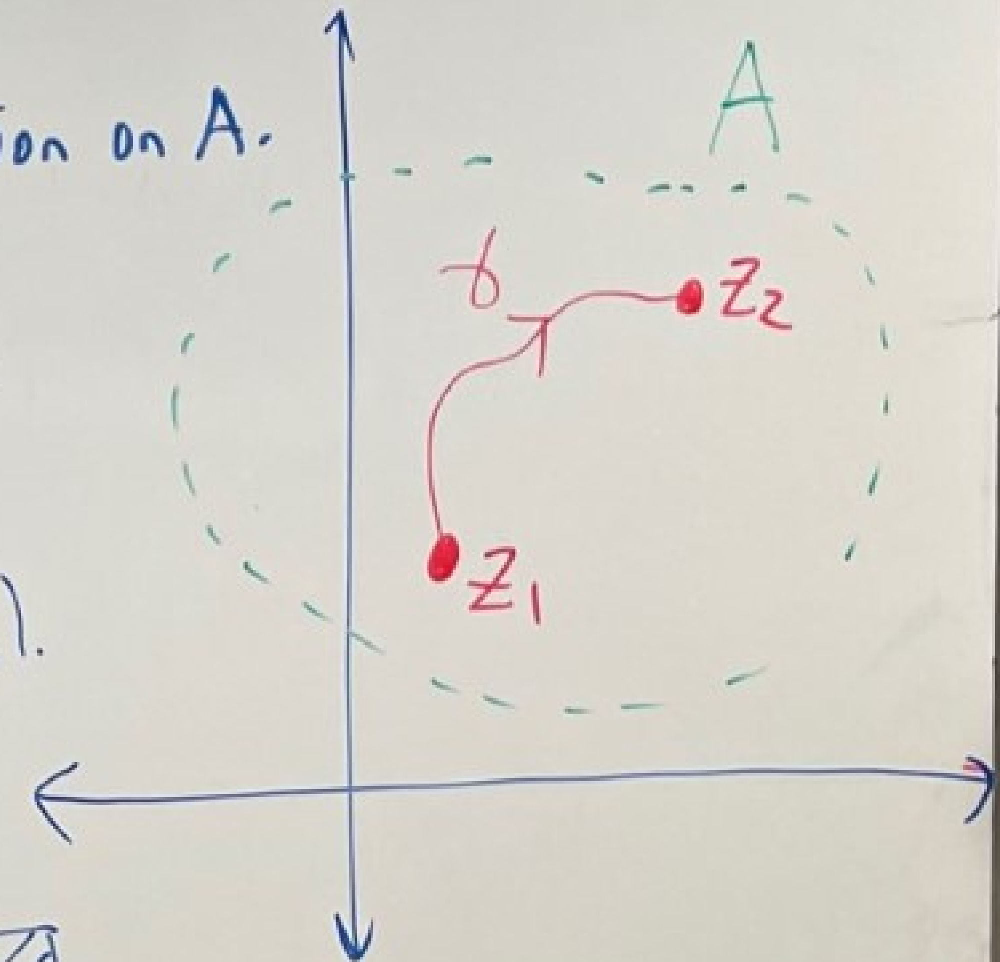
<u>proof:</u> Let $z_1, z_2$ be any two points in $A$. we will show $f(z_1) = f(z_2)$ and hence is a constant function on $A$.



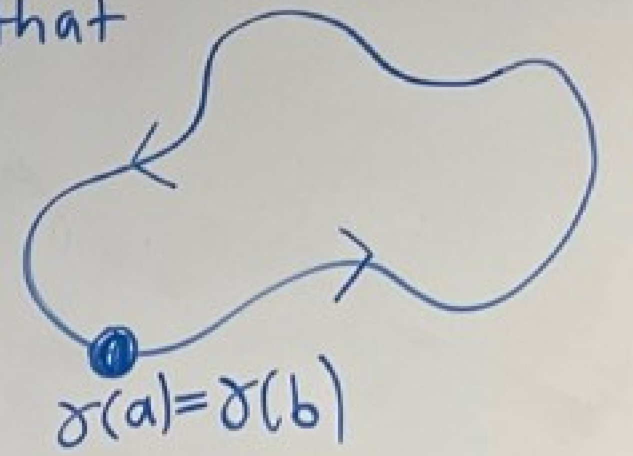

Since $A$ is path-connected there exists a piecewise smooth curve $\gamma : [a,b] \longrightarrow \mathbb{C}$ where $\gamma(a) = z_1$ and $\gamma(b) = z_2$ and $\gamma$ is contained in $A$.

Then, $$0 = \int_\gamma 0 = \int_\gamma f' \underset{\boxed{\text{FTOC}}}{=} f(\gamma(b)) - f(\gamma(a)) = f(z_2) - f(z_1).$$

So, $0 = f(z_2) - f(z_1)$.

Thus, $f(z_1) = f(z_2)$. $\blacksquare$

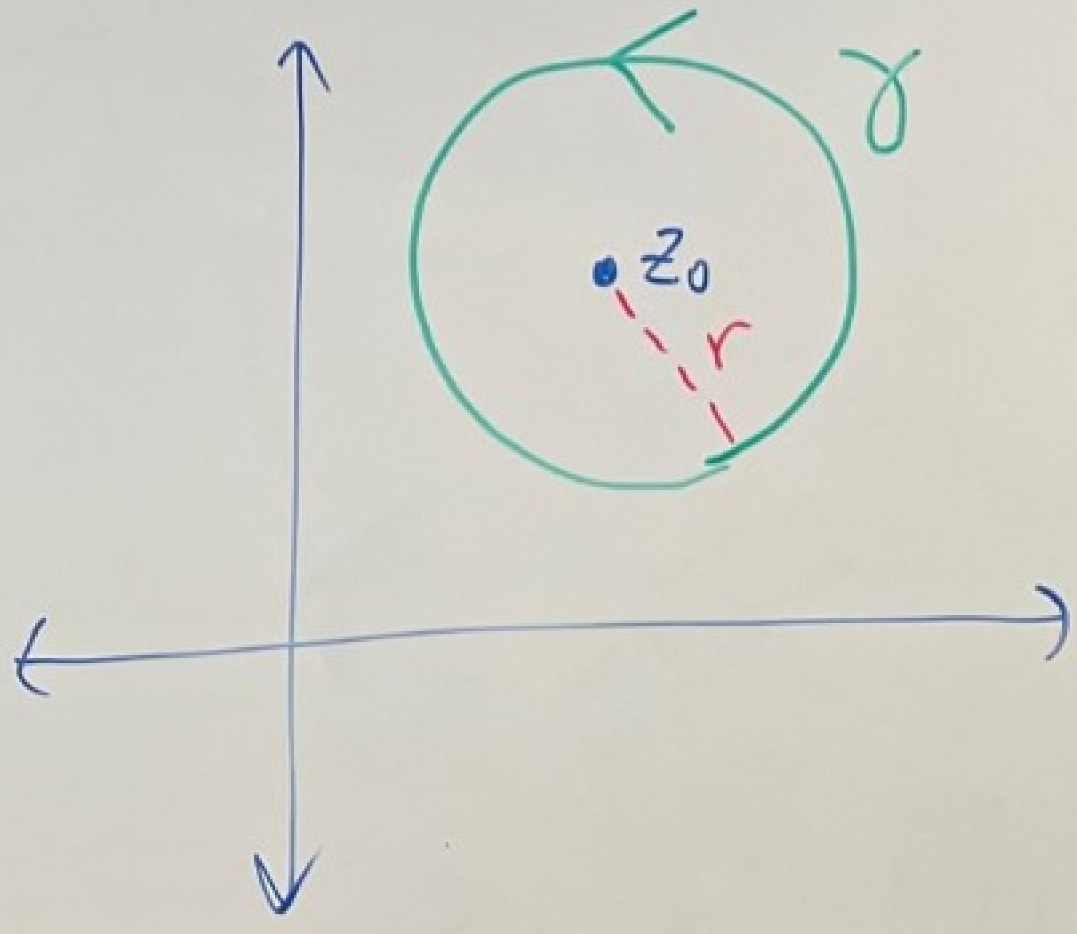
**Def:** Let $\gamma:[a,b]\to\mathbb{C}$ be a curve. We say that $\gamma$ is closed if $\gamma(a)=\gamma(b)$.




**Ex:** Suppose $\gamma$ parameterizes the circle centered at $z_0$ with radius $r$ oriented counterclockwise.

Let $n$ be any integer. Then:

$$\int_\gamma (z-z_0)^n\,dz = \begin{cases} 0 & \text{if } n\neq -1 \\ 2\pi i & \text{if } n=-1 \end{cases}$$



proof: We break this into 3 cases. Suppose $\gamma : [a,b] \to \mathbb{C}$.

case 1: Suppose $n \geq 0$. Let $F(z) = \frac{1}{n+1}(z-z_0)^{n+1}$ is analytic on all of $\mathbb{C}$ and hence $\gamma$.

And, $F'(z) = (z-z_0)^n$ is continuous on all of $\mathbb{C}$ and hence $\gamma$.

So by FTOC,

$$\int_\gamma \underbrace{(z-z_0)^n}_{F'} = F(\gamma(b)) - F(\gamma(a)) = 0$$ since $\gamma$ is a closed curve.

other cases next time...