Ex: $z_n = \frac{1}{n} + i\left(\frac{n^2-n}{n^2+1}\right)$, $n \geq 1$

$z_1 = 1 + 0i = 1$

$z_2 = \frac{1}{2} + \frac{2}{5}i$

$z_3 = \frac{1}{3} + \frac{3}{5}i$

$\vdots \qquad \vdots$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\left(\frac{1}{n} + i\left(\frac{n^2-n}{n^2+1}\right)\right)$$

thm from Monday

$$= \lim_{n\to\infty}\frac{1}{n} + i\lim_{n\to\infty}\frac{n^2-n}{n^2+1}$$

divide top/bottom by n^2

$$= 0 + i\lim_{n\to\infty}\frac{1-\frac{1}{n}}{1+\frac{1}{n^2}} = 0 + i\left(\frac{1-0}{1+0}\right)$$

$$= i$$

So, $z_n \to i$ as $n \to \infty$

Def: A sequence of complex numbers (z_n) is called **Cauchy** if for every $\varepsilon > 0$ there exists $N > 0$ where if $n, m \geq N$ then $|z_n - z_m| < \varepsilon$.

Theorem: A sequence of complex numbers (z_n) is Cauchy if and only if there exists $L \in \mathbb{C}$ where $\lim_{n\to\infty} z_n = L$.

$$\boxed{\text{Cauchy} \overset{\text{iff}}{\Longleftrightarrow} \text{converges}}$$

proof:

($\Leftarrow$) Suppose there exists $L \in \mathbb{C}$ where $\lim_{n\to\infty} z_n = L$.

Let $\varepsilon > 0$.

Since $\lim_{n\to\infty} z_n = L$ there exists $N > 0$ where if $n \geq N$ then $|z_n - L| < \frac{\varepsilon}{2}$

So if $n, m \geq N$ then

$$|z_n - z_m| = |z_n - L + L - z_m| \overset{\triangle}{\leq} |z_n - L| + |L - z_m|$$

$$= |z_n - L| + |z_m - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\boxed{|-w| = |w|}$

Thus, if $n, m \geq N$ then $|z_n - z_m| < \varepsilon$. So, (z_n) is Cauchy.

($\Rightarrow$) Suppose (z_n) is a Cauchy sequence.

Let $z_n = x_n + i y_n$.

By HW 8 #3 since (z_n) is Cauchy, this implies that (x_n) and (y_n) are Cauchy sequences in $\mathbb{R}$.

By 4650 (by using the completeness of $\mathbb{R}$), since (x_n) and (y_n) are Cauchy they must converge to real numbers.

That is, there exist real numbers X and Y where

$$\lim_{n\to\infty} x_n = X \quad \text{and} \quad \lim_{n\to\infty} y_n = Y.$$

By a $\underbrace{\text{previous theorem}}_{\text{on Monday}}$ this implies that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = \underbrace{X + iY}_{L}.$$

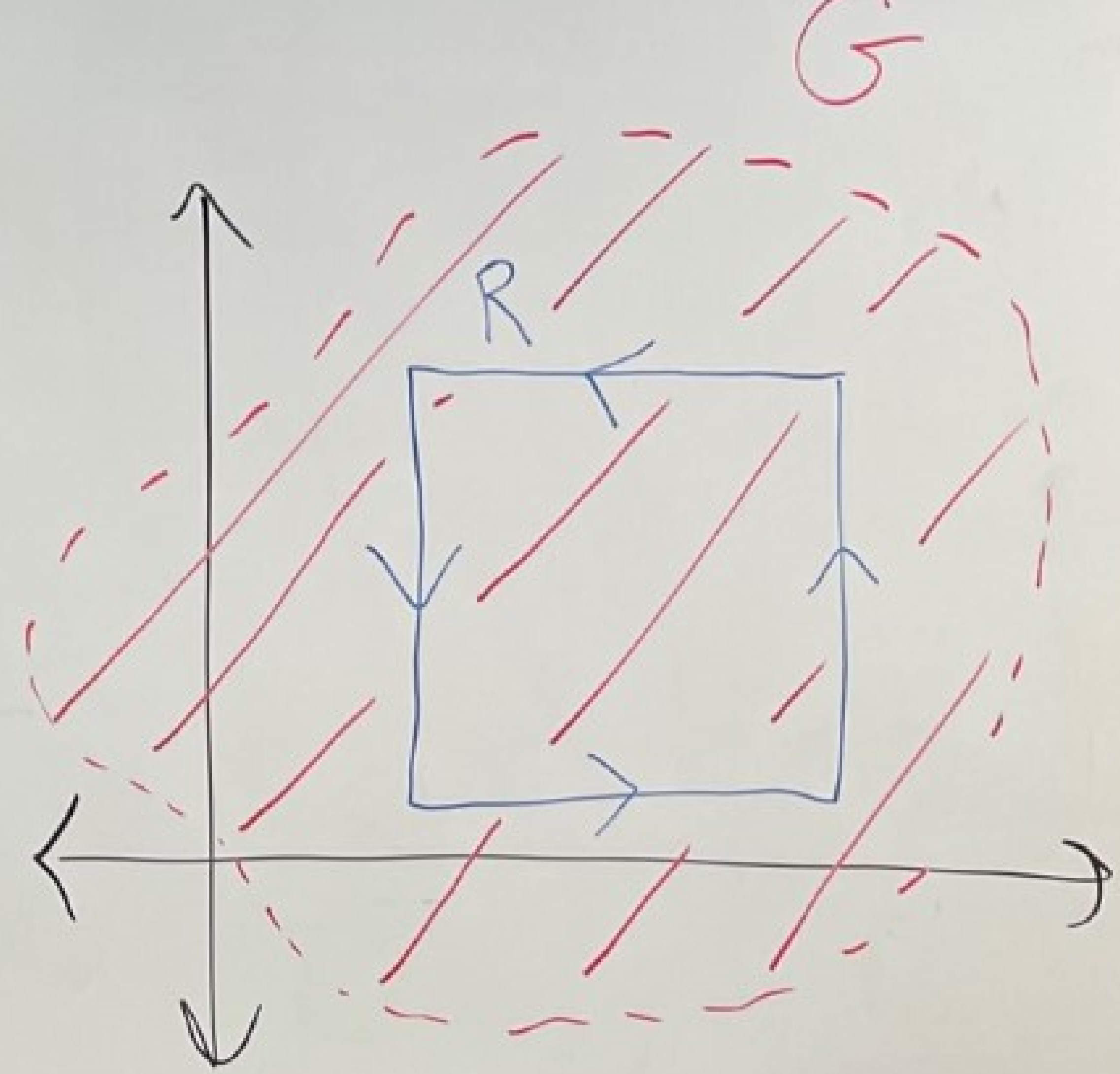
$\blacksquare$

Topic 9 – Cauchy's Theorem

Theorem: (Cauchy's theorem for a rectangle)
Suppose that R is a rectangular path with sides parallel to the xy-axes and that f is a function defined and analytic on an open set G containing R and it's interior.

Then, $\int_R f = 0$.

R is oriented counterclockwise



Proof: Next Monday

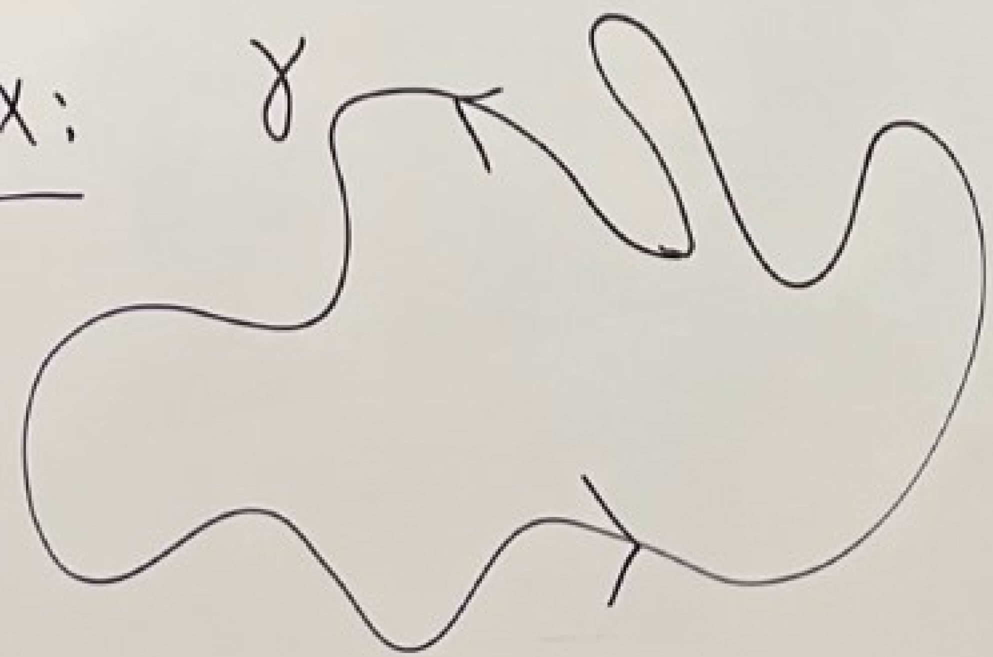
Let's generalize Cauchy's theorem.

Def: Let $\gamma : [a,b] \to \mathbb{C}$ be a closed curve ← [closed means: $\gamma(a) = \gamma(b)$]

is called simple if only the initial and ending values of γ are the same.

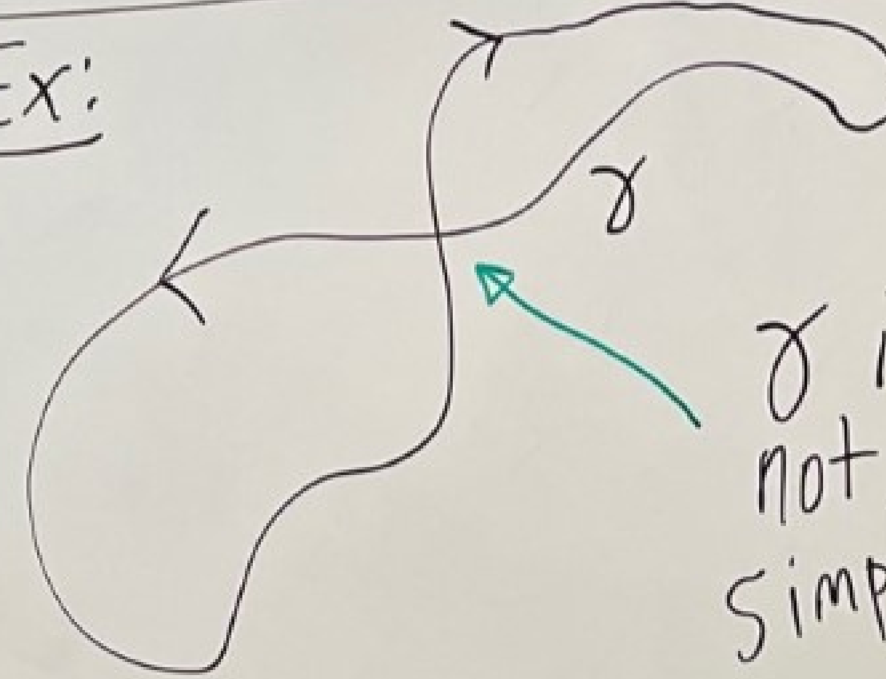
That is, if $\gamma(t_1) = \gamma(t_2)$ where $a \le t_1, t_2 \le b$, then $t_1, t_2 \in \{a, b\}$

Ex:



γ is simple

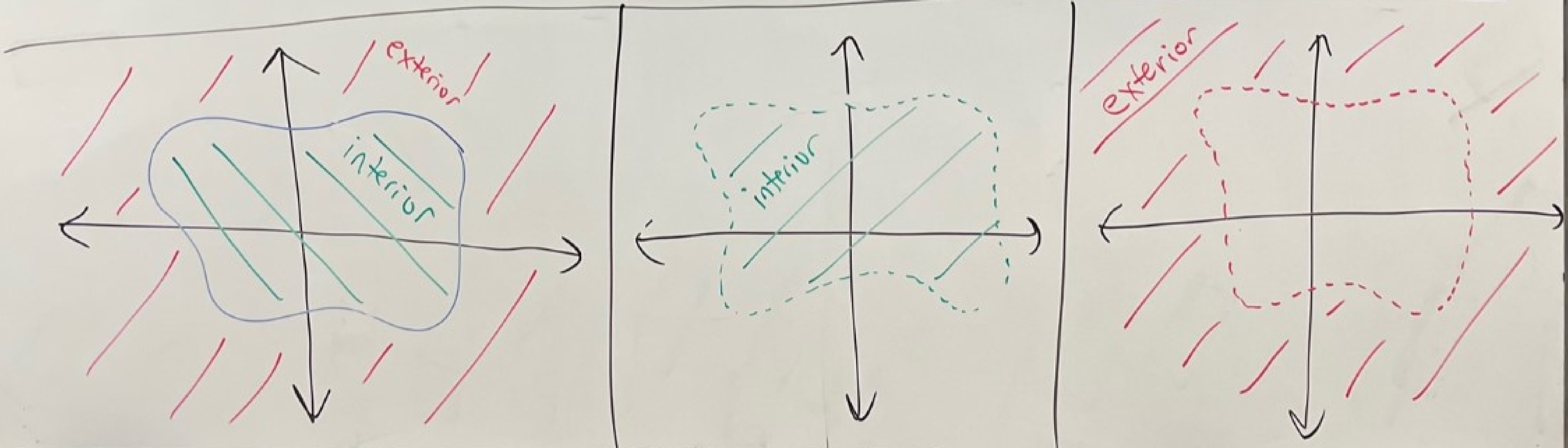
Ex:



γ is not simple

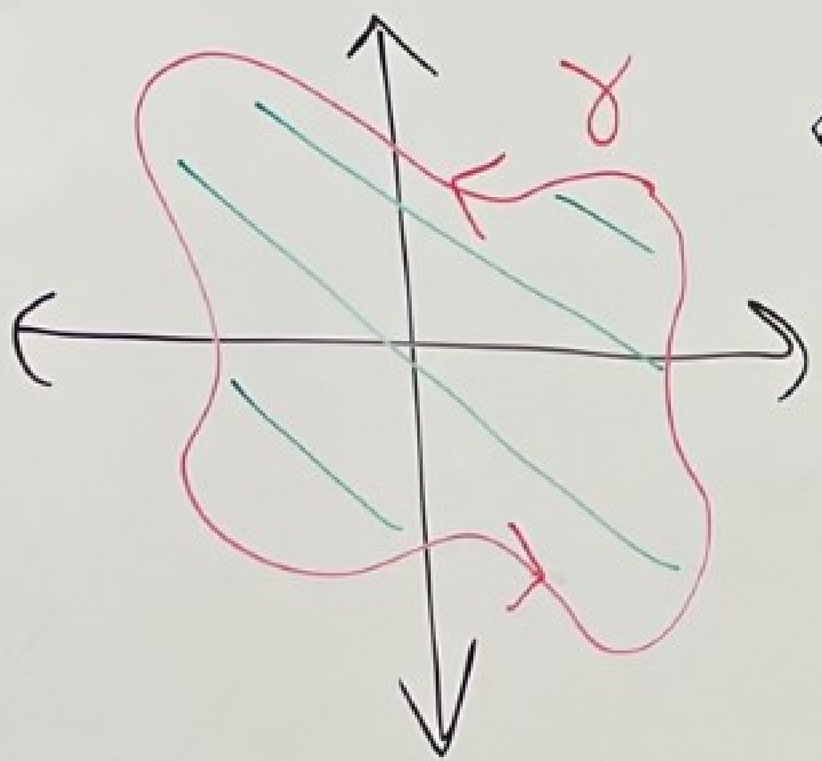
Jordan Curve Theorem

Every simple closed curve in the complex plane divides the plane into two disjoint open sets. One set (the interior of the curve) is open and bounded and the other (the exterior of the curve) is open and unbounded.



Theorem: (Cauchy's Theorem)

Let γ be a simple, closed, piecewise smooth curve.
Let f be a function that is analytic on γ and it's interior.
Then, $\int_{\gamma} f = 0$.



$\leftarrow$ f is analytic on γ and its interior

Note: Being analytic at a point means being analytic in an open set around that point.
So, being analytic on γ and its interior means being analytic on an open set containing γ and its interior.