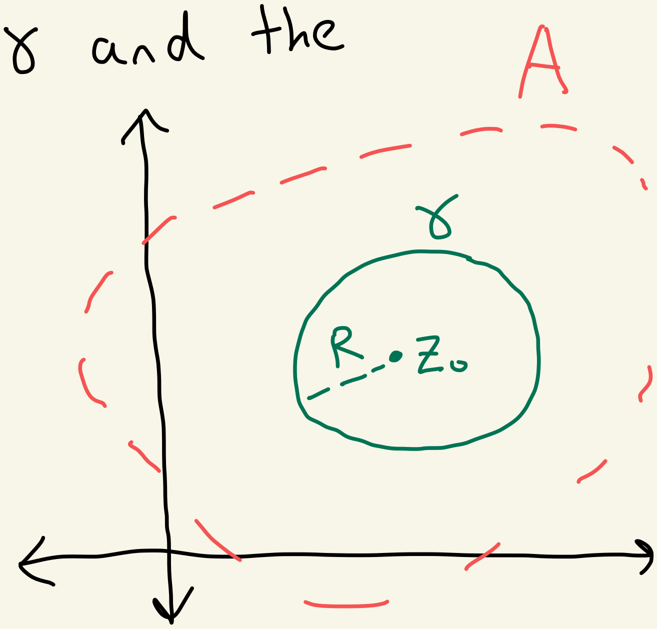
Math 4680

11/30/22

Theorem (Cauchy's Inequality)

Let f be analytic on a region A and let γ be a circle with radius $R>0$ and center $z_0 \in A$, so that γ and the interior of γ lie in A.



Suppose there exists $M>0$ where

$$|f(z)| \leq M$$

for all z on γ.

Then,
$$|f^{(k)}(z_0)| \leq \frac{k!}{R^k} \cdot M$$

for $k = 0, 1, 2, 3, \ldots$

<u>proof:</u> Orient γ counter-clockwise.
Then by the Cauchy-Integral formula

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{k+1}} dz$$

If z is on γ, then

$$\left|\frac{f(z)}{(z-z_0)^{k+1}}\right| = \frac{|f(z)|}{|z-z_0|^{k+1}} \underset{\substack{z \text{ is on } \gamma \\ |z-z_0|=R}}{=} \frac{|f(z)|}{R^{k+1}} \underset{\substack{|f(z)| \\ \text{on } \gamma}}{\leq} \frac{M}{R^{k+1}}$$

So,

$$|f^{(k)}(z_0)| = \left|\frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz\right|$$

$$\underset{|i|=1}{=} \frac{k!}{2\pi}\left|\int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz\right|$$

$$\leq \frac{k!}{2\pi}\cdot\frac{M}{R^{k+1}}\cdot\underbrace{\text{arclength}(\gamma)}_{2\pi R}$$

$$= \frac{k!}{R^k}\cdot M$$

∎

Liouville's Theorem

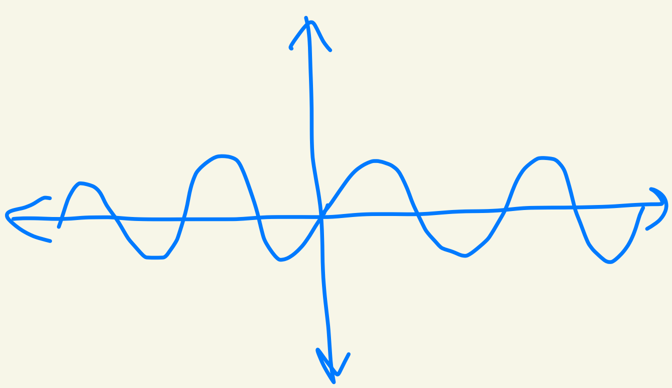
Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function that is bounded on $\mathbb{C}$. [This means that $f'(z)$ exists for all $z \in \mathbb{C}$, and there exists $M > 0$ where $|f(z)| \leq M$ for all $z \in \mathbb{C}$]

Then f is a constant function.

$\boxed{\text{So, the only bounded entire functions are the constant functions!}}$

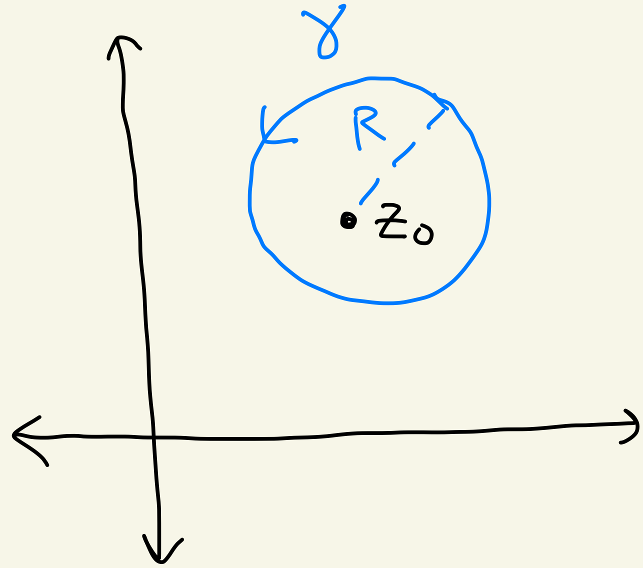
This is different from $\mathbb{R}$.

For example, $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = \sin(x)$



Then $f'(x)$ exists for all $x \in \mathbb{R}$ and $|f(x)| \leq 1$ for all $x \in \mathbb{R}$, but f is not constant.

proof: Let $f'(z)$ exist for all $z \in \mathbb{C}$.
Let $M > 0$ where $|f(z)| \leq M$ for all $z \in \mathbb{C}$.
We will show $f' = 0$ everywhere.



Let $z_0 \in \mathbb{C}$.
Let γ be a circle of radius $R > 0$ centered at z_0.
By Cauchy's inequality,

$$|f'(z_0)| \leq \frac{1!}{R^1} \cdot M = \frac{M}{R} \quad (*)$$

$\boxed{k=1}$

$(*)$ above is true for any $R > 0$. Let $R \to \infty$, then $\frac{M}{R} \to 0$.

Thus, $|f'(z_0)| = 0$.

So, $f'(z_0) = 0$.

Since $f'(z_0) = 0$ for any $z_0 \in \mathbb{C}$, this implies that f is a constant function (by a result we proved after FTOC). ∎

Fundamental theorem of Algebra

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$

where $a_0, a_1, \dots, a_n \in \mathbb{C}$, $n \geq 1$, and $a_n \neq 0$.

Then, $P(z)$ has at least one zero in the complex plane. That is, there exists $z_0 \in \mathbb{C}$ where $p(z_0) = 0$.

__proof:__ We prove this by contradiction.

Suppose $P(z) \neq 0$ for all $z \in \mathbb{C}$.

Let $f(z) = \dfrac{1}{P(z)} = \dfrac{1}{a_0 + a_1 z + \dots + a_n z^n}$.

Since $P(z) \neq 0$ for all $z \in \mathbb{C}$, we know f is an entire function.

We will now show that f is bounded on $\mathbb{C}$.

Let $w = \dfrac{a_0}{z^n} + \dfrac{a_1}{z^{n-1}} + \dfrac{a_2}{z^{n-2}} + \dots + \dfrac{a_{n-1}}{z}$.

Then, $P(z) = (a_n + w) z^n$.

Note that if $|z| \geq R$ then

$$|w| = \left| \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \dots + \frac{a_{n-1}}{z} \right|$$

$$\overset{\triangle}{\leq} \left| \frac{a_0}{z^n} \right| + \left| \frac{a_1}{z^{n-1}} \right| + \left| \frac{a_2}{z^{n-2}} \right| + \dots + \left| \frac{a_{n-1}}{z} \right|$$

$|z| \geq R$

$\frac{1}{|z|} \leq \frac{1}{R}$

$$\leq \frac{|a_0|}{R^n} + \frac{|a_1|}{R^{n-1}} + \frac{|a_2|}{R^{n-2}} + \dots + \frac{|a_{n-1}|}{R}$$

Note that $\frac{|a_i|}{R^{n-i}} \longrightarrow 0$ as $R \to \infty$ (for $0 \leq i \leq n-1$).

So we may pick $R > 0$ big enough so that

$$\frac{|a_i|}{R^{n-i}} < \frac{|a_n|}{2n}$$

$\frac{|a_n|}{2n}$ is a fixed positive real #

for all $0 \leq i \leq n-1$.

Fix such an $R > 0$.

Then if $|z| \geq R$ we have

$$|w| \leq \frac{|a_0|}{R^n} + \frac{|a_1|}{R^{n-1}} + \frac{|a_2|}{R^{n-2}} + \dots + \frac{|a_{n-1}|}{R}$$

$$< \frac{|a_n|}{2n} + \frac{|a_n|}{2n} + \frac{|a_n|}{2n} + \dots + \frac{|a_n|}{2n}$$

$$= n\left(\frac{|a_n|}{2n}\right) = \frac{|a_n|}{2}$$

So if $|z| \geq R$, then

$$|a_n + w| \geq \big||a_n| - |w|\big| = |a_n| - |w| > |a_n| - \frac{|a_n|}{2} = \frac{|a_n|}{2}$$

because $|w| < \frac{|a_n|}{2} < |a_n|$

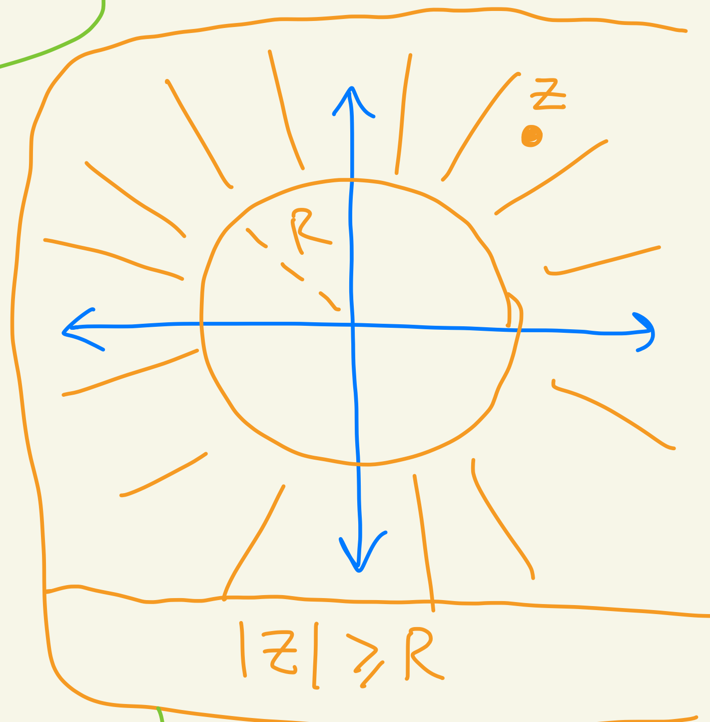
So, $|a_n| - |w| > 0$

$|w| < \frac{|a_n|}{2}$

$-|w| > -\frac{|a_n|}{2}$

Thus, if $|z| \geq R$, then

$$|P(z)| = |a_n + w||z^n| > \frac{|a_n|}{2} \cdot |z|^n \geq \frac{|a_n|}{2} \cdot R^n$$

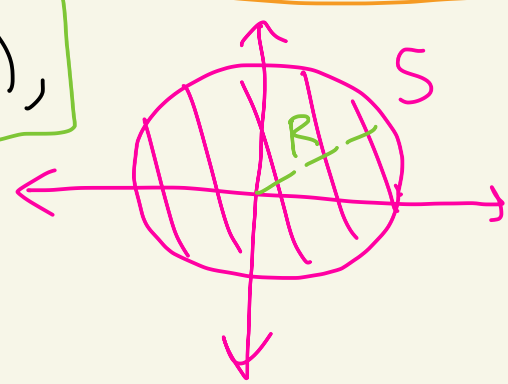


Thus, if $|z| \geq R$, then

$$|f(z)| = \frac{1}{|P(z)|} < \frac{2}{|a_n| \cdot R^n}$$

By analysis (4650) / topology (4690) results since f is continuous on $S = \{z \mid |z| \leq R\}$ and S is closed and bounded (ie compact), then f is bounded on S.

That is, there exists $K > 0$ where $|f(z)| \leq K$ if $|z| \leq R$.



Let $M = \max\left\{ \frac{2}{|a_n| \cdot R^n}, K \right\}$.

Then, $|f(z)| \leq M$ for all $z \in \mathbb{C}$.

So, f is entire and bounded, thus by Louiville's theorem, $f(z) = c$ for some $c \in \mathbb{C}$.

But then $P(z) = \frac{1}{f(z)} = \frac{1}{c}$ for all $z \in \mathbb{C}$.

But $P(z) = a_0 + a_1 z + \dots + a_n z^n$ is not a constant function since $n \geq 1$ and $a_n \neq 0$.

Contradiction!

Thus, there must exist at least one zero of $P(z)$ in $\mathbb{C}$. $\blacksquare$