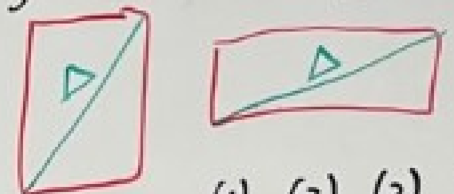
(Cauchy's thm on a rectangle)

$f: G \to \mathbb{C}$, G is open containing rectangle R, f analytic on $G \implies \int_R f = 0$

pf: Orient R in the counterclockwise direction.

Let P be the perimeter of R [ie arclength of R].

Let Δ be the length of the main diagonal.



Divide R into four congruent smaller rectangles $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$.

If each subrectangle is oriented counter-clockwise the cancellation along common edges gives us that

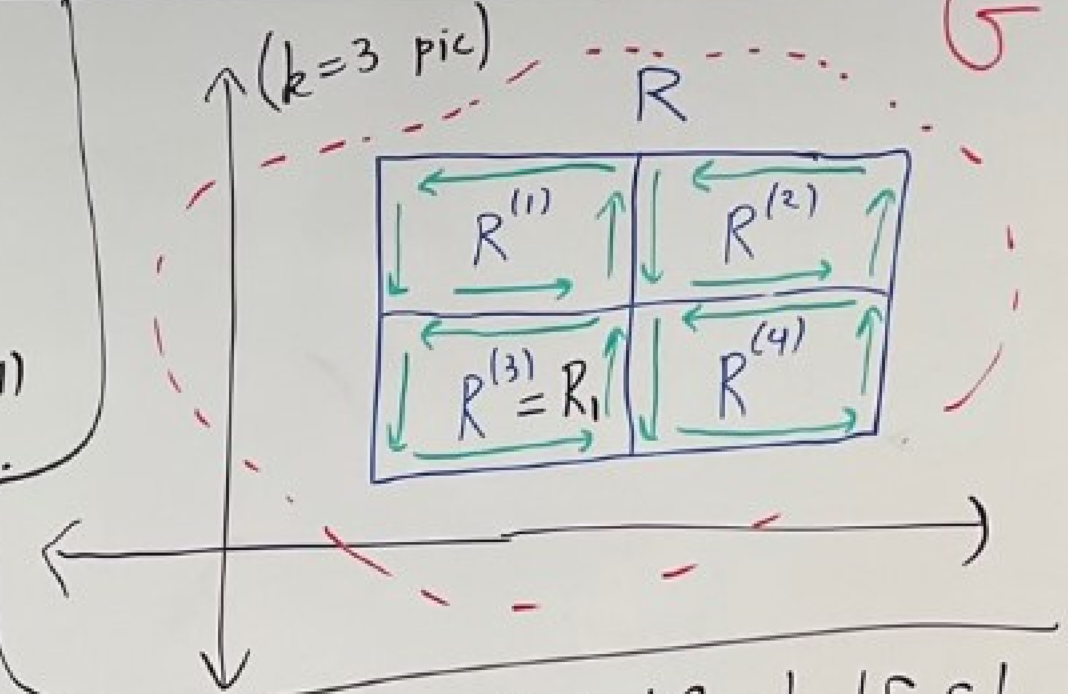
$$\int_R f = \int_{R^{(1)}} f + \int_{R^{(2)}} f + \int_{R^{(3)}} f + \int_{R^{(4)}} f.$$

Triangle inequality gives

$$\left|\int_R f\right| \le \left|\int_{R^{(1)}} f\right| + \left|\int_{R^{(2)}} f\right| + \left|\int_{R^{(3)}} f\right| + \left|\int_{R^{(4)}} f\right|.$$

Thus, for one of these rectangles, call it $R^{(k)}$, we must have $\left|\int_{R^{(k)}} f\right| \ge \frac{1}{4}\left|\int_R f\right|$.

Call this $R^{(k)}$ by R_1.



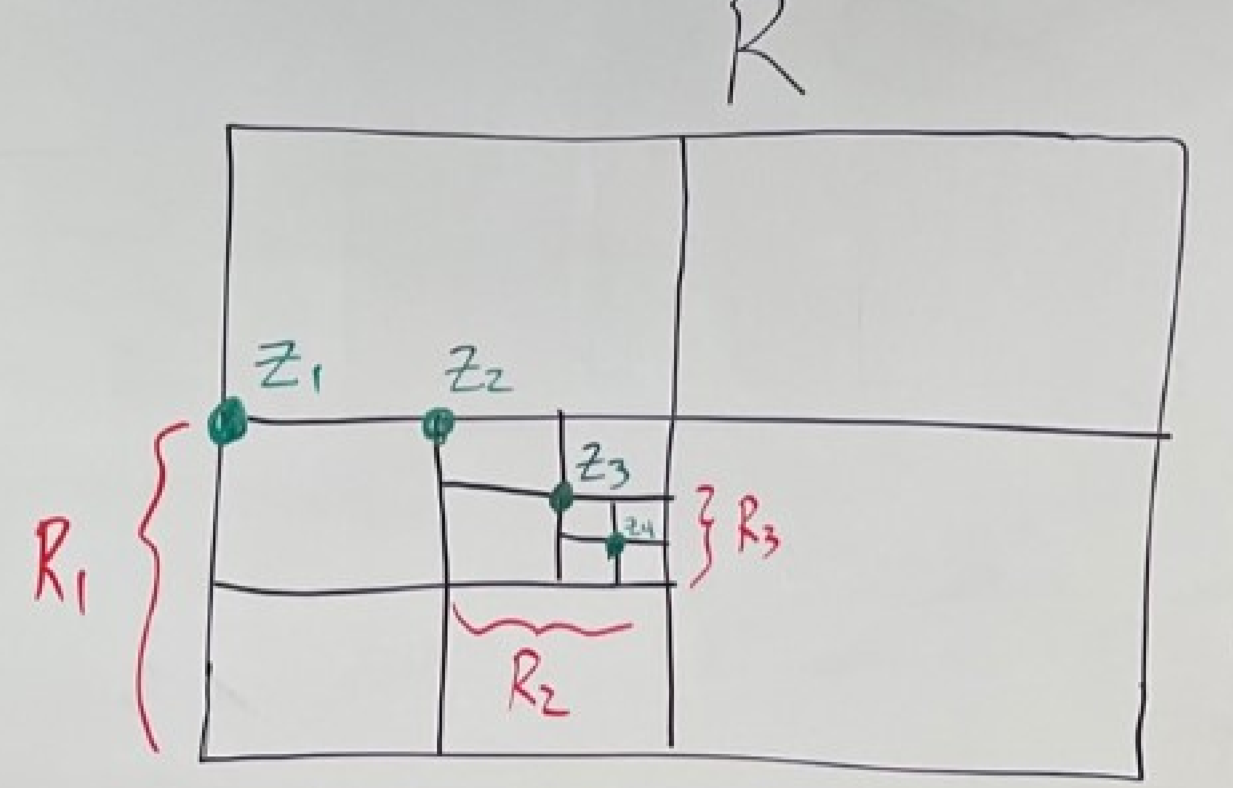
Notice that the perimeter and diagonal of R_1 are half those of R.

Now repeat this bisection process inside of R_1, obtaining a sequence $R_1, R_2, R_3, \dots$ of smaller and smaller rectangles where:

(i) $\left|\int_{R_n} f\right| \geq \frac{1}{4}\left|\int_{R_{n-1}} f\right| \geq \frac{1}{4^2}\left|\int_{R_{n-2}} f\right| \geq \dots \geq \frac{1}{4^n}\left|\int_R f\right|$

(ii) $\text{Perimeter}(R_n) = \frac{1}{2^n}\,\text{perimeter}(R) = \frac{P}{2^n}$

(iii) $\text{Diagonal}(R_n) = \frac{1}{2^n}\,\text{Diagonal}(R) = \frac{\Delta}{2^n}$



Let z_n be the upper-left corner of R_n.

Claim: (z_n) is a Cauchy sequence.

pf of claim: Let $\varepsilon > 0$.

Choose $N > 0$ so that $\underbrace{\frac{\Delta}{2^N}}_{\text{main diagonal of } R_N} < \varepsilon$.

Thus, if $n, m \geq N$ then

$$|z_n - z_m| \leq \text{diagonal}(R_N) = \frac{\Delta}{2^N} < \varepsilon.$$

since $n, m \geq N$ z_n and z_m are in or on R_N

$\boxed{\text{Claim}}$

Therefore there exists $w_0 \in \mathbb{C}$ with $\lim_{n \to \infty} z_n = w_0$.

Let $\varepsilon > 0$ be fixed for the remainder of the proof.

Our goal will be to show that $\left|\int_R f\right| \leq \varepsilon \Delta P$.

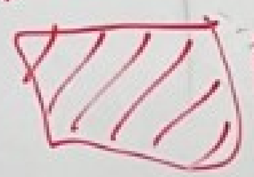
We need some facts.

length of main diagonal of R_n

<u>Fact 0:</u> If z is on or inside R_n, then $|z - w_0| \leq \dfrac{\Delta}{2^n}$.

<u>pf of Fact 0:</u> If $k \geq n$, then z_k is on or inside R_n.

Since $\underbrace{R_n \cup (\text{inside of } R_n)}$ is a closed set and $\lim_{k\to\infty} z_k = w_0$ by HW 8 #6


R_n

this implies that w_0 is on or inside R_n. Since z and w_0 are both in or on R_n, we have $|z - w_0| \leq \dfrac{\Delta}{2^n}$. $\boxed{\text{Fact 0}}$

Fact 1: w_0 lies on or in R.

pf: We showed in Fact 0 that w_0 is on or in R_n for all $n \geq 1$.

$\boxed{\text{Fact } 1}$

Fact 2: Since w_0 is on or in R, we know that $f'(w_0)$ exists.

So, $\displaystyle\lim_{z \to w_0} \frac{f(z)-f(w_0)}{z-w_0} = f'(w_0)$.

Thus there exists $\delta > 0$ where if $0 < |z-w_0| < \delta$, then

$$\left| \frac{f(z)-f(w_0)}{z-w_0} - f'(w_0) \right| < \varepsilon.$$ Thus, if $|z-w_0| < \delta$, then

$$\left| f(z) - f(w_0) - f'(w_0)(z-w_0) \right| \leq \varepsilon |z-w_0|.$$

Let $\widehat{N}$ be large enough so that $\frac{\Delta}{2^n} < \delta$ if $n \geq \widehat{N}$.

So if $n \geq \widehat{N}$ and z is on or in R_n, then $|z - w_0| \leq \frac{\Delta}{2^n} < \delta$

$\boxed{\text{Fact 0}}$

So if $n \geq \widehat{N}$ and z is on or in R_n, then

$$|f(z) - f(w_0) - (z - w_0) f'(w_0)| \leq \varepsilon |z - w_0| \leq \varepsilon \frac{\Delta}{2^n}.$$

$\boxed{\text{Fact 2}}$

<u>Fact 3:</u> By FTOC, $\int_{R_n} 1 \, dz = 0$ and $\int_{R_n} (z - w_0) \, dz = 0$.

$\boxed{\text{Fact 3}}$

Thus, if $n \geq \widehat{N}$ we have that

$$\left|\int_R f\right| \leq 4^n \left|\int_{R_n} f\right| \underset{\text{Fact 3}}{=} 4^n \left| \int_{R_n} f(z)\,dz - f(w_0)\overbrace{\int_{R_n} 1\,dz}^{0} - f'(w_0)\overbrace{\int_{R_n}(z-w_0)\,dz}^{0}\right|$$

$$= 4^n\left|\int_{R_n}\big(f(z)-f(w_0)-f'(w_0)(z-w_0)\big)\,dz\right|$$

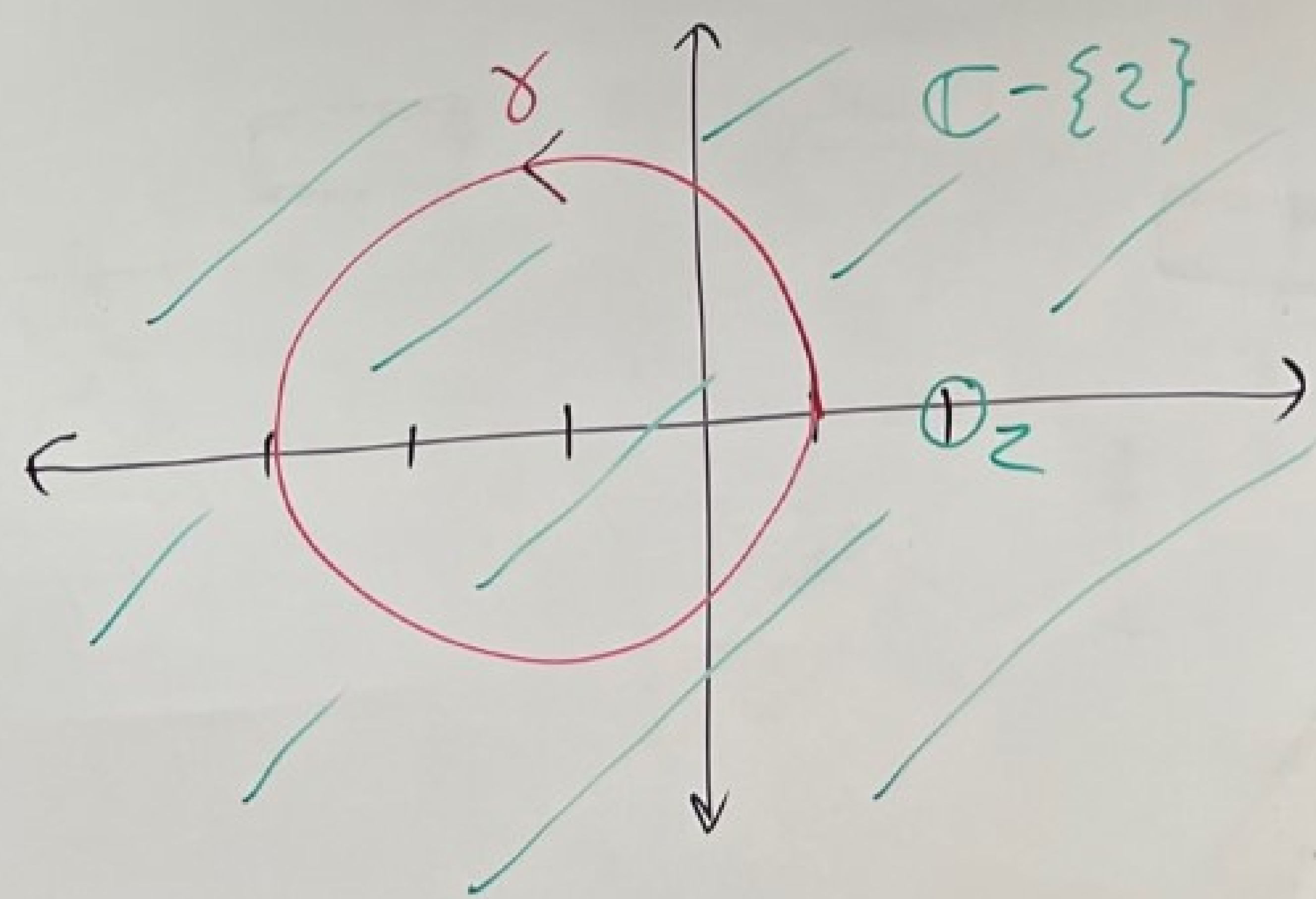
$$\underset{\text{Fact 2}}{\leq} 4^n\left(\frac{\varepsilon\Delta}{2^n}\right)\cdot\underbrace{\text{arclength}(R_n)}_{\text{perimeter}(R_n)} = 4^n\left(\frac{\varepsilon\Delta}{2^n}\right)\left(\frac{P}{2^n}\right) = \varepsilon\Delta P.$$

Thus, for every $\varepsilon>0$, $\left|\int_R f\right| \leq \varepsilon\underbrace{\Delta P}_{\text{fixed}}$. So, $\left|\int_R f\right|=0$. Thus, $\int_R f = 0$ ∎

Conway

? Complex Analysis

<u>Ex:</u> Let γ be the circle centered at -1 with radius 2, oriented counter-clockwise.



Compute $\displaystyle\int_\gamma \frac{1}{z-2}\,dz$.

$\frac{1}{z-2}$ is analytic on $\mathbb{C}-\{2\}$.

So, $\frac{1}{z-2}$ is analytic on and inside γ.

By the more general Cauchy thm,

$$\int_\gamma \frac{1}{z-2}\,dz = 0$$