4680

12/5/22

Lemma: Suppose f is analytic on a region A and $|f(z)|$ is constant on A. Then $f(z)$ is constant on A.

proof: Suppose that $f(x+iy) = u(x,y) + iv(x,y)$. We are assuming that for all $x+iy \in A$ we have

$$|f(x+iy)|^2 = \left(\sqrt{u(x,y)^2 + v(x,y)^2}\right)^2 = (u(x,y))^2 + (v(x,y))^2 = c$$

for some constant $c \in \mathbb{R}$, $c \geq 0$.

If $c = 0$, then $|f(x+iy)| = 0$ for all $x+iy \in A$.
Then, $f(x+iy) = 0$ for all $x+iy \in A$.
Then f is constant on A.
So we can now assume $c \neq 0$.
We know $(u(x,y))^2 + (v(x,y))^2 = c$ on A.
Differentiating we get

$$\boxed{\begin{aligned} 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} &= 0 \\ 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} &= 0 \end{aligned}} \quad (*)$$

Since f is analytic on A we know $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ on A.

Sub these into (*) and divide by 2 to get

$$\boxed{\begin{aligned} u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} &= 0 \\ v\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} &= 0 \end{aligned}} \qquad (**)$$

on A.

Then (**) becomes

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix}\begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (***)$$

For any fixed input (x,y) the above linear system has two equations and two "unknowns."

Since $\det\begin{pmatrix} u & -v \\ v & u \end{pmatrix} = u^2 + v^2 = c \neq 0$

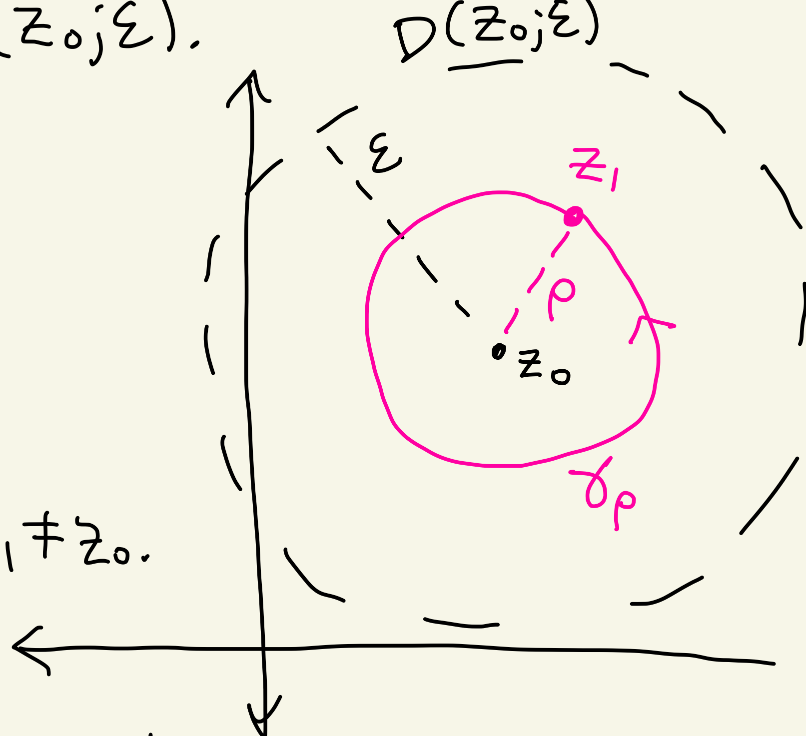
there is only one unique solution to (***) which is $\frac{\partial u}{\partial x}(x,y) = \frac{\partial u}{\partial y}(x,y) = 0$.

Thus, if $x+iy \in A$ then

$$\begin{aligned} f'(x+iy) &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) \\ &= \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y) \\ &= 0 - i\,0 = 0 \end{aligned}$$

So, $f' = 0$ on the region A. By a previous thm, f is constant on A.

<u>Theorem:</u> (Special case of max modulus theorem).
Suppose that f is analytic on $D(z_0;\varepsilon)$ where $z_0 \in \mathbb{C}$ and $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.
If $|f(z)| \leq |f(z_0)|$ for all $z \in D(z_0;\varepsilon)$, then f is constant on $D(z_0;\varepsilon)$.



<u>proof:</u> Suppose
$|f(z)| \leq |f(z_0)|$
for all $z \in D(z_0;\varepsilon)$.

Let $z_1 \in D(z_0;\varepsilon)$ where $z_1 \neq z_0$.
Let $\rho = |z_1 - z_0|$
Let γ_ρ be the circle centered at z_0 with radius ρ, oriented counterclockwise.

By the Cauchy-integral theorem

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(z)}{z - z_0}\, dz$$

Parameterize γ_ρ as $\gamma_\rho(t) = z_0 + \rho e^{it}$ where $0 \leq t \leq 2\pi$. And $\gamma_\rho'(t) = i\rho e^{it}$.

So we get

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0+\rho e^{it})}{(z_0+\rho e^{it}) - z_0} \cdot i\rho e^{it} \, dt \qquad (*)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0+\rho e^{it}) \, dt$$

From $(*)$ we get

$$|f(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0+\rho e^{it}) \, dt \right|$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0+\rho e^{it})| \, dt$$

I put a proof of this method of bounding in the topic 11 notes on page 30

Since $|f(z_0+\rho e^{it})| \leq |f(z_0)|$ for all t by assumption we get that

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0+\rho e^{it})| \, dt \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| \, dt$$

from above

$$= \frac{1}{2\pi} \left[|f(z_0)| \cdot (2\pi - 0) \right] = |f(z_0)|$$

Thus, $|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{it})|\,dt$

So, $\underbrace{\frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,dt}_{|f(z_0)|} = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{it})|\,dt$

Thus, $\frac{1}{2\pi}\int_0^{2\pi} \Big[\underbrace{|f(z_0)| - |f(z_0+\rho e^{it})|}_{\geq 0}\Big]\,dt = 0$

We are integrating a continuous function that is ≥ 0 and the integral equals 0.

The only way this can happen is if

$$|f(z_0)| - |f(z_0+\rho e^{it})| = 0$$

for all t.

So, $|f(z_0)| = |f(\underbrace{z_0+\rho e^{it}}_{f(\text{pt on } \gamma_\rho)})|$ for all t.

In particular, $|f(z_0)| = |f(z_1)|$

Since z_1 was arbitrary, $|f(z_0)| = |f(z)|$ for all $z \in D(z_0;\varepsilon)$.

So, $|f(z)|$ is constant on $D(z_0;\varepsilon)$.

By the lemma, f is constant on $D(z_0;\varepsilon)$. $\blacksquare$

<u>Theorem:</u> (Max modulus theorem)
Suppose that f is analytic on a region A and f is <u>not</u> constant on A.
Then f does not have a maximum value on A.
That is, there does <u>not</u> exist $z_0 \in A$ where $|f(z)| \leq |f(z_0)|$ for all $z \in A$.

<u>proof:</u> Churchill / Brown book
maybe in Hoffman / Marsden book. $\blacksquare$