Let $z, z_0 \in \mathbb{C}$.

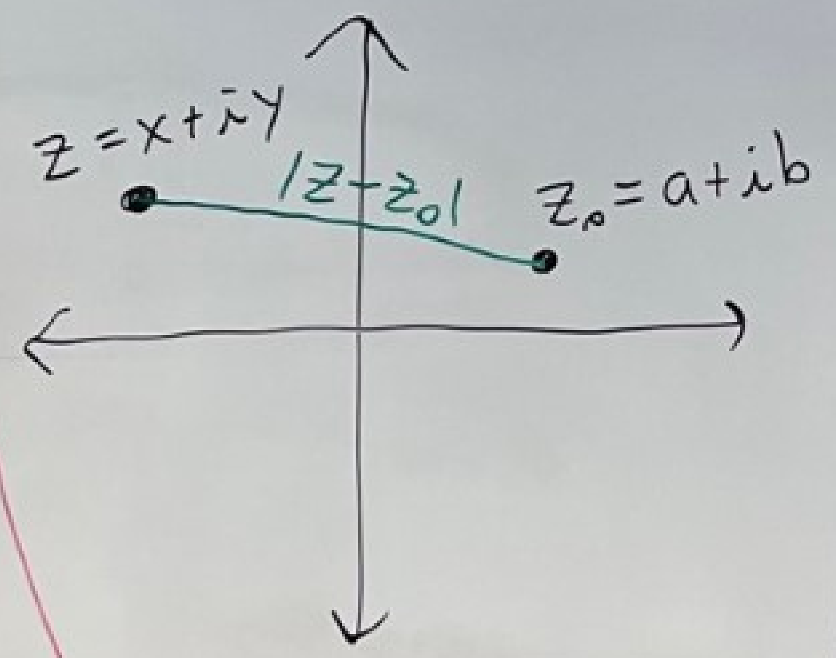
Suppose $z = x + iy$ and $z_0 = a + ib$. Then,

$$|z - z_0| = |(x-a) + i(y-b)|$$

$$= \sqrt{(x-a)^2 + (y-b)^2}$$

distance between (x,y) and (a,b)

or between $x+iy$ and $a+ib$



HW 1 6(c)

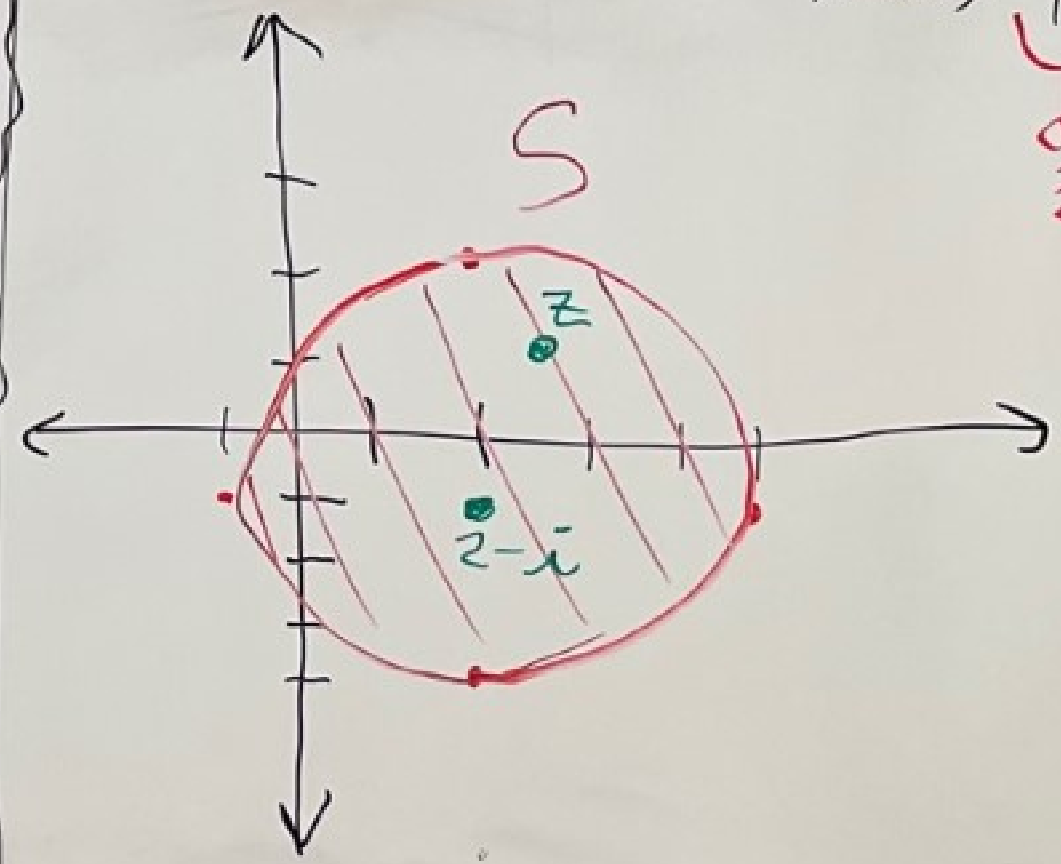
Graph

$$S = \{ z \in \mathbb{C} \mid |z - 2 + i| \leq 3 \}$$

Suppose $|z - 2 + i| \leq 3$

Then, $|z - (2 - i)| \leq 3$

distance between z and $2 - i$



HW 1 6(a)

Graph $S = \{ z \in \mathbb{C} - \{0\} \mid \operatorname{Re}(\frac{1}{z}) \geq \frac{1}{2} \}$

Let $z = x + iy$.

Then, $\boxed{\operatorname{Re}(\frac{1}{z}) \geq \frac{1}{2}}$

iff $\operatorname{Re}\left(\frac{1}{x+iy}\right) \geq \frac{1}{2}$

iff $\operatorname{Re}\left(\frac{1}{x+iy} \cdot \frac{x-iy}{x-iy}\right) \geq \frac{1}{2}$

iff $\operatorname{Re}\left(\frac{x-iy}{x^2+y^2}\right) \geq \frac{1}{2}$

iff $\operatorname{Re}\left(\left(\frac{x}{x^2+y^2}\right) + i\left(\frac{-y}{x^2+y^2}\right)\right) \geq \frac{1}{2}$

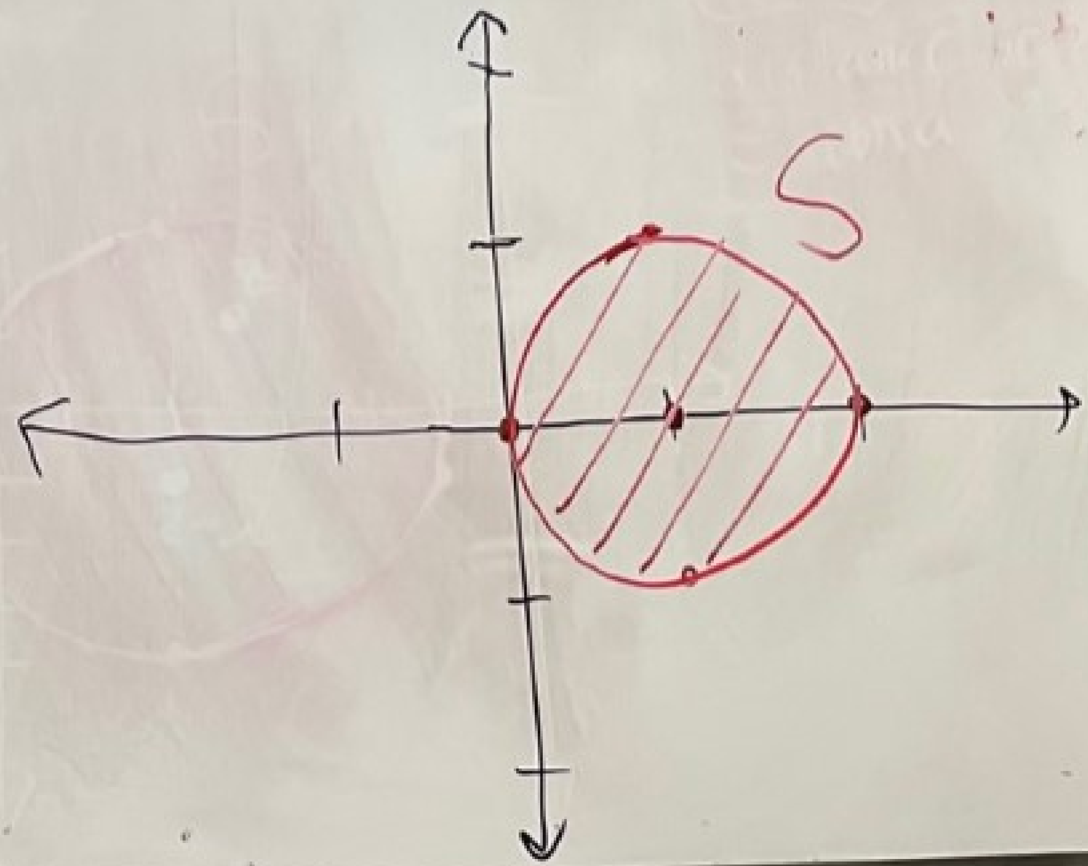
iff $\frac{x}{x^2+y^2} \geq \frac{1}{2}$

iff $2x \geq x^2 + y^2$

iff $x^2 - 2x + y^2 \leq 0$

iff $x^2 - 2x + 1 + y^2 \leq 1$

iff $\boxed{(x-1)^2 + (y-0)^2 \leq 1}$



Def:
$$e^{x+iy} = e^x(\cos(y) + i\sin(y))$$

Proposition:

1. $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$
2. $|e^{x+iy}| = e^x$ for all $x, y \in \mathbb{R}$
3. $e^z \neq 0$ for all $z \in \mathbb{C}$
4. e^z is $2\pi i$-periodic, that is
$$e^{z+2\pi i k} = e^z$$
for all integers k.

proof:

1. Let $z = x+iy$ and $w = a+ib$

Then,

$$e^{z+w} = e^{(x+a)+i(y+b)}$$
$$\overset{\text{def}}{=} e^{x+a}[\cos(y+b) + i\sin(y+b)]$$
$$= e^{x+a}\big[\cos(y)\cos(b) - \sin(y)\sin(b) + i[\cos(y)\sin(b) + \cos(b)\sin(y)]\big]$$
$$= e^x e^a [\cos(y) + i\sin(y)][\cos(b) + i\sin(b)]$$

(real e valued)

$$= e^x(\cos(y)+i\sin(y))\, e^a(\cos(b)+i\sin(b))$$

$$\overset{\text{def}}{=} e^{x+iy} e^{a+ib}$$

$$= e^z e^w$$

② $$|e^{x+iy}| = |e^x(\cos(y)+i\sin(y))| \overset{|uv|=|u||v|}{=} |e^x|\,|\cos(y)+i\sin(y)| = e^x\sqrt{(\cos(y))^2+(\sin(y))^2}$$

$$= e^x\sqrt{1} = e^x$$

($e^x > 0$; $\cos(y)+i\sin(y)$ on unit circle)

③ Let $z = x+iy$.

Then $|e^z| \overset{②}{=} e^x \neq 0$ (real-valued e^x)

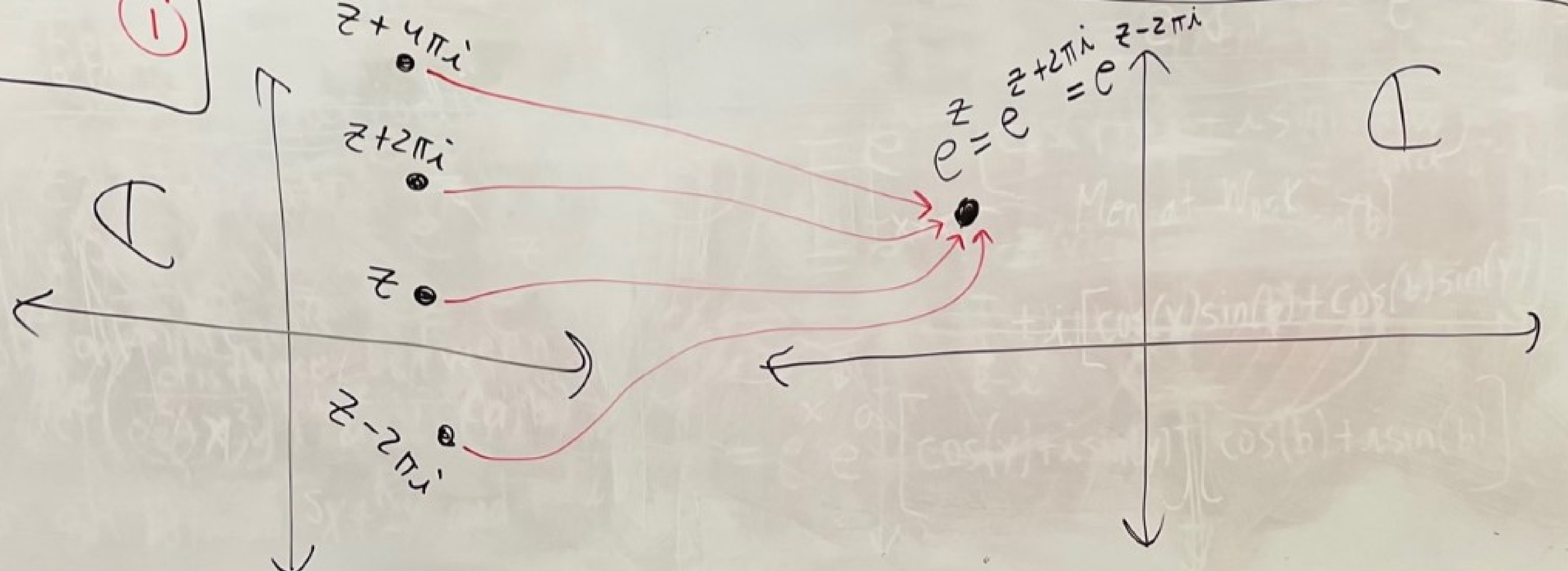
(4) Let $z \in \mathbb{C}$ and $k \in \mathbb{Z}$

$$e^{0+2\pi i k} = e^{0}\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

$e^{0} = 1$ (1)

Then,

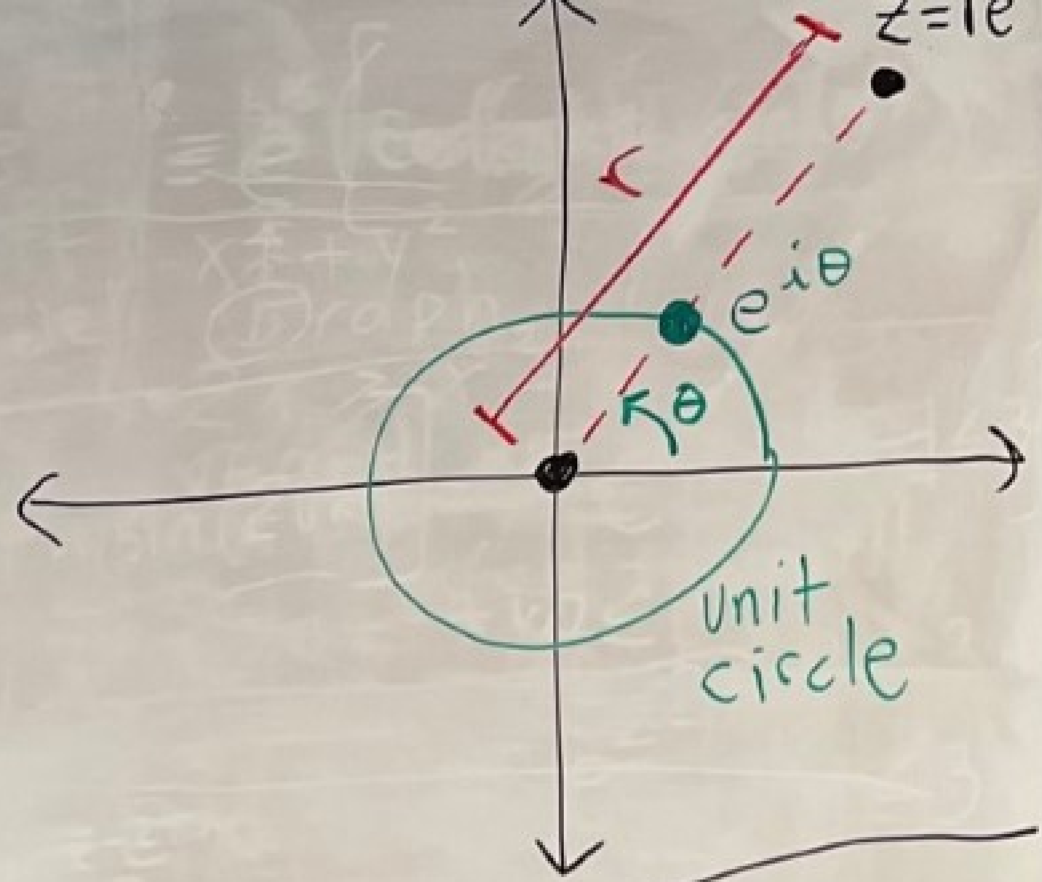
$$e^{z+2\pi i k} \overset{(1)}{=} e^{z} e^{2\pi i k} = e^{z} e^{i(2\pi k)} = e^{z}\left[\underbrace{\cos(2\pi k)}_{1} + i\underbrace{\sin(2\pi k)}_{0}\right] = e^{z}$$



Note: (Polar form of z)

Suppose $z = r\left[\cos(\theta) + i\sin(\theta)\right] = re^{i\theta}$

($\cos(\theta) + i\sin(\theta)$: on unit circle)



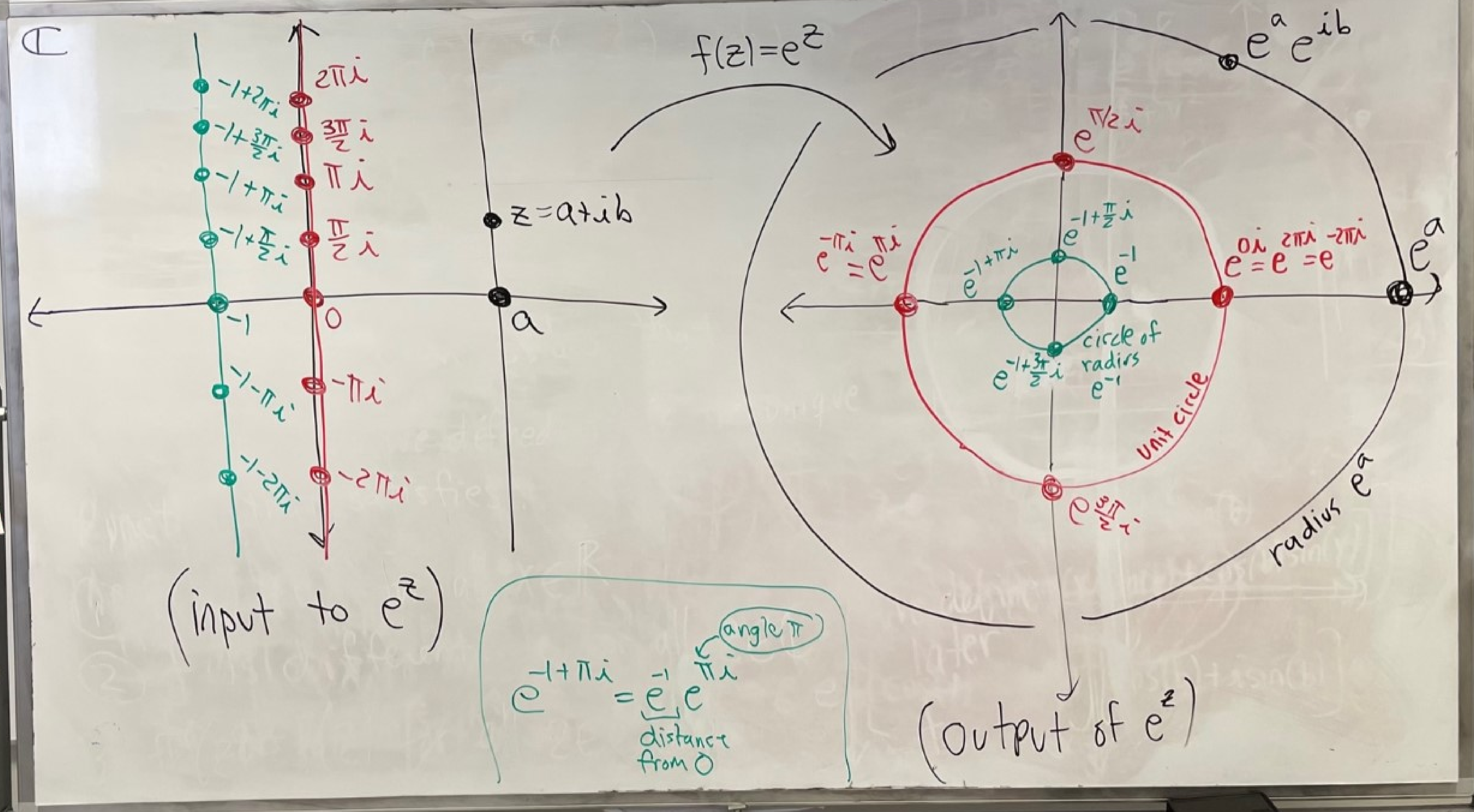
Note: One can show that $f(z) = e^{z}$ (as we defined it) is the unique function that satisfies:

① $f(x) = e^{x}$ for all $x \in \mathbb{R}$

② f is differentiable for all $z \in \mathbb{C}$

③ $f'(z) = f(z)$ for all $z \in \mathbb{C}$

We define f' later


ℂ
$-1+2\pi i$
$-1+\frac{3\pi}{2}i$
$-1+\pi i$
$-1+\frac{\pi}{2}i$
-1
$-1-\pi i$
$-1-2\pi i$
$2\pi i$
$\frac{3\pi}{2}i$
πi
$\frac{\pi}{2}i$
0
$-\pi i$
$-2\pi i$
$z=a+ib$
a
(input to e^z)
$f(z)=e^z$
$e^a e^{ib}$
$e^{\pi/2\, i}$
$e^{-1+\frac{\pi}{2}i}$
$e^{-\pi i}=e^{\pi i}$
$e^{-1+\pi i}$
e^{-1}
$e^{0i}=e^{2\pi i}=e^{-2\pi i}$
e^a
circle of radius e^{-1}
$e^{-1+\frac{3\pi}{2}i}$
unit circle
$e^{\frac{3\pi}{2}i}$
radius e^a
$e^{-1+\pi i}=e^{-1}e^{\pi i}$
angle π
distance from 0
(output of e^z)